COMPUTING L-FUNCTIONS AND SEMISTABLE REDUCTION OF SUPERELLIPTIC CURVES

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Abstract. We give an explicit description of the stable reduction of superelliptic curves of the form $y^n = f(x)$ at primes $\mathfrak{p}$ whose residue characteristic is prime to the exponent n. We then use this description to compute the local L-factor of the curve and the exponent of conductor at $\mathfrak{p}$.

1. Introduction

1.1. Let Y be a smooth projective curve of genus $g \geq 2$ over a number field K. The *L-function* of Y is defined as an Euler product

$$L(Y,s) := \prod_{\mathfrak{p}} L_{\mathfrak{p}}(Y,s),$$

where $\mathfrak{p}$ ranges over the prime ideals of K. The local L-factor $L_{\mathfrak{p}}(Y,s)$ is defined as follows. Choose a decomposition group $D_{\mathfrak{p}} \subset \mathrm{Gal}(\bar{K}/K)$ of $\mathfrak{p}$. Let $I_{\mathfrak{p}} \subset D_{\mathfrak{p}}$ denote the inertia subgroup and let $\sigma_{\mathfrak{p}} \in D_{\mathfrak{p}}$ be an arithmetic Frobenius element (i.e. $\sigma_{\mathfrak{p}}(\alpha) \equiv \alpha^{\mathrm{N}\mathfrak{p}} \pmod{\mathfrak{p}}$). Then

$$L_{\mathfrak{p}}(Y,s) := \det\left(1 - \mathrm{N}\mathfrak{p}^{-s}\sigma_{\mathfrak{p}}^{-1}|V^{I_{\mathfrak{p}}}\right)^{-1},$$

where

$$V := H^1_{\mathrm{et}}(Y \otimes_K \bar{K}, \mathbb{Q}_\ell)$$

is the first étale cohomology group of Y (for some auxiliary prime ℓ distinct from the residue characteristic of $\mathfrak{p}$).

Another arithmetic invariant of Y closely related to $L(Y,s)$ is the *conductor of the L-function*. Similar to $L(Y,s)$, it is defined as a product over local factors (times a power of the discriminant of K):

$$N := \delta_K^{2g} \cdot \prod_{\mathfrak{p}} (\mathrm{N}\mathfrak{p})^{f_{\mathfrak{p}}},$$

where $f_{\mathfrak{p}}$ is a nonnegative integer called the *exponent of conductor* at $\mathfrak{p}$. The integer $f_{\mathfrak{p}}$ measures the *ramification* of the Galois module V at the prime $\mathfrak{p}$. See [19], §2 for a precise definition.

Many spectacular conjectures and theorems concern these L-functions. For instance, it is conjectured that $L(Y,s)$ has a meromorphic continuation to the entire complex plane, and a functional equation of the form

$$\Lambda(Y,s) = \pm\Lambda(Y, 2-s), \tag{1}$$

where

$$\Lambda(Y,s) := N^{s/2}(2\pi)^{-gs}\Gamma(s)^g L(Y,s).$$

This conjecture can be proved for certain special curves related to automorphic forms (like modular curves) and, as a consequence of the Taniyama-Shimura conjecture, for elliptic curves over $\mathbb{Q}$. Besides that, very little is known.

1.2. One motivation for this paper is the question: how can we compute the defining series for $L(Y,s)$ and the conductor N explicitly for a given curve Y? By definition, this is a local problem at each prime ideal $\mathfrak{p}$. So we fix $\mathfrak{p}$ and aim at computing $L_{\mathfrak{p}}(Y,s)$ and $f_{\mathfrak{p}}$. Note that the residue field of $\mathfrak{p}$ is the finite field $\mathbb{F}_q$ with $q = \mathrm{N}\mathfrak{p}$ elements.

Assume first that Y has *good reduction* at $\mathfrak{p}$. This means that there exists a flat proper model $\mathcal{Y}$ of Y over $\mathcal{O}_K$ whose special fiber $\bar{Y} = \bar{Y}_{\mathfrak{p}}$ at $\mathfrak{p}$ is smooth. Standard theorems in étale cohomology show that the action of $\mathrm{Gal}(\bar{K}/K)$ on $V = H^1_{\mathrm{et}}(Y_{\bar{K}}, \mathbb{Q}_\ell)$ is unramified at $\mathfrak{p}$ (i.e. $I_{\mathfrak{p}}$ acts trivially) and therefore the exponents of conductor vanishes, $f_{\mathfrak{p}} = 0$. Furthermore, the local L-factor $L_{\mathfrak{p}}(Y,s)$ is equal to the inverse of the denominator of the zeta function of $\bar{Y}$, i.e.

$$Z(\bar{Y}, q^{-s}) = \frac{L_{\mathfrak{p}}(Y,s)^{-1}}{(1-q^{-s})(1-q^{1-s})},$$

where

$$Z(\bar{Y},T) := \exp\left(\sum_{n\geq 1} |\bar{Y}(\mathbb{F}_{q^n})| \cdot \frac{T^n}{n}\right)$$

To compute $L_{\mathfrak{p}}(Y,s)$ for small prime ideals we simply need to count the number of $\mathbb{F}_{q^n}$-rational points on $\bar{Y}$, for $n = 1,\ldots,g$.

If Y has bad reduction it is much harder to compute $L_{\mathfrak{p}}(Y,s)$ and $f_{\mathfrak{p}}$. To our knowledge, there are essentially three ways to proceed:

(1) Compute a *regular model* of Y at $\mathfrak{p}$.
(2) Compute the *stable reduction* of Y at $\mathfrak{p}$.
(3) Guess the local L-factors at *all* primes of bad reduction, and then verify your guess via the functional equation for $L(Y,s)$.

All three methods have certain advantages and drawbacks, and it is often a combination of them which works best. In this paper we would like to advertise method (2), by demonstrating its simplicity and power in a large class of examples.

1.3. Before we get into more details of method (1) and (2), let us briefly describe method (3). Let $\mathfrak{p}_1,\ldots,\mathfrak{p}_r$ be the prime ideals of the number field K where Y has bad reduction. One can show the following:

- For $i = 1,\ldots,r$ there are only finitely many possible choices for the local L-factor $L_{\mathfrak{p}_i}(Y,s)$ and the exponent $f_{\mathfrak{p}_i}$. In fact, the set of all choices depends only on the norm $q_i = \mathrm{N}\mathfrak{p}_i$ and the genus g.

- There is at most a unique choice of the conductor N and the local L-factors $L_{\mathfrak{p}_i}(Y,s)$ at the bad primes $\mathfrak{p}_i$ such that the L-function
$$L(Y,s) := \prod_{\mathfrak{p}} L_{\mathfrak{p}}(Y,s)$$
satisfies the functional equation (1).

This suggests the following strategy to determine $L(Y,s)$:

- Guess the conductor $N = \prod_i q_i^{f_i}$ and the local L-factors $L_{\mathfrak{p}_i}(Y,s)$ at the bad primes $\mathfrak{p}_i$.
- Compute $L_{\mathfrak{p}}(Y,s)$ for all good primes $\mathfrak{p}$ with $\mathrm{N}\mathfrak{p} \leq C$ for some sufficiently large constant C. If C is not too large, this can be done by simple point counting.
- Check numerically whether $L(Y,s) := \prod_{\mathfrak{p}} L_{\mathfrak{p}}(Y,s)$ satisfies the functional equation (1). By [6], we need to choose $C \sim N^{1/2}$.

In practice, this can be done if $N \sim 10^{15}$. See e.g. [7].

An obvious drawback of this method is that one can never prove that the guess one has made is correct.

1.4. Regular models. Fix a prime ideal $\mathfrak{p}$ of K. Since the local L-factor $L_{\mathfrak{p}}(Y,s)$ and the exponent $f_{\mathfrak{p}}$ depend only on the base change of Y to the completion $\hat{K}_{\mathfrak{p}}$, we may and will from now on assume that K is a p-adic local field, i.e. a finite extension of $\mathbb{Q}_p$. Let $\mathbb{F}_q$ denote the residue field of K and q the number of elements of $\mathbb{F}_q$.

We may assume that Y has bad reduction. By resolution of singularities of two-dimensional schemes, there exists a *regular model* $\mathcal{Y}^{\mathrm{reg}}$, i.e. a flat and proper $\mathcal{O}_K$-model of Y which is regular. Since we assume $g \geq 2$ we may also assume that $\mathcal{Y}^{\mathrm{reg}}$ is the minimal regular model. Let $\bar{Y}^{\mathrm{reg}}$ denote the special fiber of $\mathcal{Y}^{\mathrm{reg}}$. Under an additional (relatively mild) assumption, it is still true that $L_{\mathfrak{p}}(Y,s)$ is the inverse of the denominator of the zeta function of the special fiber $\bar{Y}^{\mathrm{reg}}$ of $\mathcal{Y}^{\mathrm{reg}}$ (as in the smooth case). (See Proposition 2.6 below.) Therefore, $L_{\mathfrak{p}}(Y,s)$ can be computed from $\bar{Y}^{\mathrm{reg}}$ by point counting.

By a result of Saito ([16]) it should also be possible to compute $f_{\mathfrak{p}}$ from $\mathcal{Y}^{\mathrm{reg}}$, but we are not aware of any attempt to do this explicitly for nonhyperelliptic curves of genus $g \geq 2$.

Finding a regular model $\mathcal{Y}^{\mathrm{reg}}$ can be computationally challenging. The computer algebra system MAGMA has a build-in function to compute regular models of curves of genus $g \geq 2$, but it seems that there are still many restrictions on the types of curves for which it works. A similar function which should overcome these limitations is being prepared in SINGULAR.

1.5. Semistable reduction. The alternative to regular models are semistable models. By a theorem of Deligne and Mumford ([5]) there exists a finite extension L/K such that the curve $Y_L := Y \otimes_K L$ has semistable reduction. More precisely: there exists a model $\mathcal{Y}^{\mathrm{ss}}$ of Y_L whose special fiber $\bar{Y}^{\mathrm{ss}}$ is reduced with at most ordinary double points as singularities.

As with regular models, our assumption $g \geq 2$ guarantees that there is a minimal semistable model called the *stable model*. The special fiber $\bar{Y}^{\rm ss}$ of the stable model $\mathcal{Y}^{\rm ss}$ is called the *stable reduction* of Y. We may also assume that the extension L/K is Galois. Then the Galois group $\Gamma := {\rm Gal}(L/K)$ has a natural *semilinear* action on $\mathcal{Y}^{\rm ss}$ (i.e. an action lifting the tautological action on $\operatorname{Spec}\mathcal{O}_L$). Restricting this action to the special fiber we obtain a natural, semilinear action of Γ on $\bar{Y}^{\rm ss}$. Let $\bar{Z}_{\mathbb{F}_q} := \bar{Y}^{\rm ss}/\Gamma$ denote the quotient scheme. Note that $\bar{Z}_{\mathbb{F}_q}$ is a reduced but possibly singular curve over the finite field $\mathbb{F}_q$. We have the following result (which is certainly known to experts, but not so easy to find in the literature).

Theorem 1.1. *The stable reduction $\bar{Y}^{\rm ss}$, together with its natural Γ-action, determines the local L-factor $L_{\mathfrak{p}}(Y,s)$ and the exponent $f_{\mathfrak{p}}$. In particular, $L_{\mathfrak{p}}(Y,s)^{-1}$ is the denominator of the zeta function of $\bar{Z}_{\mathbb{F}_q}$ (which may be computed by point counting).*

1.6. Let us compare the two methods discussed above. If the curve Y has already semistable reduction at $\mathfrak{p}$ then the minimal regular model of Y is also semistable. In this case there is no essential difference between the two methods. In general, however, the two methods are quite different in nature.

From the theoretical point of view one may consider the method of stable reduction as 'better' because it gives more information. For instance, unlike the regular model, the stable model is invariant under base change of the curve Y to any finite extension K'/K. Therefore, once the stable reduction of Y has been computed, the desired invariants like $L_{\mathfrak{p}}(Y,s)$ and $f_{\mathfrak{p}}$ can be computed directly for any curve $Y' := Y \otimes_K K'$. It is also much easier to compute the exponent $f_{\mathfrak{p}}$ from the stable reduction (but for other invariants like the local Tamagawa number, the converse is true.)

From a computational point of view it may seem to be a lot easier to find a regular model. After all, to compute a semistable model is essentially equivalent to computing a regular model over a larger field L *and* to find the correct extension L/K in the first place.

1.7. Superelliptic curves. The results of the present paper show that there are classes of curves for which it is actually rather easy to determine the stable reduction, even though the reduction behaviour can be arbitrarily complicated. We consider *superelliptic curves*, i.e. curves Y given by an equation of the form

$$y^n = f(x),$$

where n is a positive integer and $f(x)$ is a rational function over a p-adic number field K. The additional and crucial condition we impose is that the exponent n must be prime to the residue characteristic p of K.

Let L_0/K be the splitting field of $f(x)$, i.e. the smallest extension of K over which all poles and zeroes of $f(x)$ become rational. Our main result in §4 says that Y has semistable reduction over an explicit and at most

tamely ramified extension L/L_0. Moreover, the stable reduction $\bar{Y}^{\rm ss}$, together with the natural action of $\Gamma = \mathrm{Gal}(L/K)$, can be described easily and in a purely combinatorial manner. The only part which may be computationally difficult is the analysis of the extension L_0/K. Indeed, by choosing $f(x)$ appropriately we can make this extension as large and as complicated as we want. On the other hand, it is possible to fabricate examples where the computation of the stable reduction is still rather easy, but the standard algorithms for computing a regular model fail.

Starting from the description of the stable reduction, we also given an explicit procedure to determine an equation for the quotient curve $\bar{Z}_{\mathbb{F}_q} := \bar{Y}^{\rm ss}/\Gamma$ in §5. This equation can then be used to compute the local L-factor, via Theorem 1.1 above.

We remark that our description of the stable reduction of superelliptic curves is based on a very special case of more general results on *admissible reduction* for covers of curves. These results are well known to experts. The main goal of the present paper is to make these results more widely known and to demonstrate their usefulness for explict computations.

2. Étale cohomology of curves

2.1. Let p be a prime number and K a finite extension of $\mathbb{Q}_p$. The residue field of K is a finite field with $q = p^f$ elements; we denote it by $\mathbb{F}_q$.

We choose an algebraic closure $\bar{K}$ of K and write $\Gamma_K = \mathrm{Gal}(\bar{K}/K)$ for the absolute Galois group of K. Let $K^{\rm nr} \subset \bar{K}$ be the maximal unramified extension of K and $I_K := \mathrm{Gal}(\bar{K}/K^{\rm nr})$ the inertia group of K. Then we have a short exact sequence

$$1 \to I_K \to \Gamma_K \to \Gamma_{\mathbb{F}_q} \to 1,$$

where $\Gamma_{\mathbb{F}_q} = \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ is the absolute Galois group of $\mathbb{F}_q$. Note that $\Gamma_{\mathbb{F}_q}$ is the free profinite group of rank one generated by the *Frobenius element* Frob_q, which is defined by $\mathrm{Frob}_q(\alpha) = \alpha^q$.

2.2. Let Y/K be a smooth projective and absolutely irreducible curve over K. We assume that the genus g of Y satisfies $g \geq 2$.

We fix an auxiliary prime $\ell \neq p$. We wish to describe the natural action of Γ_K on the étale cohomology group

$$V = H^1_{\rm et}(Y_{\bar{K}}, \mathbb{Q}_\ell) := \big(\varprojlim_n H^1_{\rm et}(Y_{\bar{K}}, \mathbb{Z}/\ell^n)\big) \otimes \mathbb{Q}_\ell.$$

On a more practical level, we are interested in computing certain invariants of this Galois representation, in particular the *local L-factor* and the *exponent of conductor*. The local L-factor is the function $L(Y/K, s) := P_1(Y/K, q^{-s})^{-1}$, where

$$P_1(Y/K, T) := \det(1 - \mathrm{Frob}_q^{-1} \cdot T \mid V^{I_K}).$$

The *exponent of conductor* is defined as the integer

$$f = f_{Y/K} = \epsilon + \delta,$$

where

$$\epsilon := \dim V - \dim V^{I_K}$$

is the codimension of the I_K-invariant subspace and where δ is the so called *Swan conductor* of V. See e.g. [19], §2, for a precise definition. It turns out that $f_{Y/K} = -\mathrm{Art}(Y/K)$, where $\mathrm{Art}(Y/K)$ is the *Artin conductor* of Y/K, as defined e.g. in [16].

The invariant $f_{Y/K}$ depends only on the I_K-action on V, and vanishes if the I_K-action is trivial (i.e. if V is *unramified*). In general it gives a measure of 'how bad' the ramification is.

2.3. Let $K^{\mathrm{nr}} = \bar{K}^{I_K}$ denote the maximal unramified extension of K. Let k denote the residue field of K^{nr}, which is an algebraic closure of the residue field $\mathbb{F}_q$ of K. By the theorem of Deligne and Mumford ([5]) there exists a finite extension L/K^{nr} such that the curve $Y_L := Y \otimes_K L$ has semistable reduction. This means that there exists a flat and proper $\mathcal{O}_L$-model $\mathcal{Y}$ of Y_L whose special fiber $\bar{Y} := \mathcal{Y} \otimes k$ is reduced and has at most ordinary double points as singularities.

Assume that Y has semistable reduction over L/K^{nr}. Then there exists a unique minimal semistable model $\mathcal{Y}$ of Y_L, called the *stable model*. Its special fiber $\bar{Y}$ is called the *stable reduction* of Y. It is an important point that $\bar{Y}$ is independent of the choice of the extension L/K^{nr}. As a result, we obtain a canonical action of Γ_K on $\bar{Y}$, as follows. We may assume that the extension L/K^{nr} is Galois. Then Γ_K acts on $Y_L = Y \otimes_K L$ via the second factor. The uniqueness of the stable model shows that this action extends to an action on $\mathcal{Y}$, which we may restrict to the special fiber $\bar{Y} = \mathcal{Y} \otimes k$. By construction, the resulting action of Γ_K on $\bar{Y}$ is semilinear, meaning that the structure map $\bar{Y} \to \operatorname{Spec} k$ is Γ_K-equivariant. In particular, the restriction of this action to the inertia subgroup I_K is k-linear. Moreover, it factors via the finite quotient $\mathrm{Gal}(L/K^{\mathrm{nr}})$.

When we say *stable reduction* of Y, we always mean the curve $\bar{Y}$, together with the Γ_K-action defined above. Note that for a finite extension K'/K the stable reduction of $Y_{K'}$ is the same k-curve $\bar{Y}$, together with the restriction of the Γ_K-action to its subgroup $\Gamma_{K'}$.

The main result in this section is:

Theorem 2.1. *We have a natural,* $\mathrm{Gal}(k/\mathbb{F}_q)$*-equivariant isomorphism*

$$H^1_{\mathrm{et}}(Y_{\bar{K}}, \mathbb{Q}_\ell)^{I_K} \cong H^1_{\mathrm{et}}(\bar{Y}/I_K, \mathbb{Q}_\ell).$$

Remark 2.2. The action of Γ_K on the quotient curve $\bar{Z} := \bar{Y}/I_K$ factors through the quotient $\Gamma_K \to \mathrm{Gal}(k/\mathbb{F}_q)$. The resulting $\mathrm{Gal}(k/\mathbb{F}_q)$-action on the k-curve $\bar{Z}$ defines an $\mathbb{F}_q$-structure, i.e. there exists an $\mathbb{F}_q$-curve $\bar{Z}_{\mathbb{F}_q}$ such that $\bar{Z} = \bar{Z}_{\mathbb{F}_q} \otimes k$. This means that the trace of the $\mathbb{F}_q$-Frobenius acting on $H^1_{\mathrm{et}}(\bar{Z}, \mathbb{Q}_\ell)$ (and hence the action of $\mathrm{Gal}(k/\mathbb{F}_q)$ on $H^1_{\mathrm{et}}(Y_{\bar{K}}, \mathbb{Q}_\ell)^{I_K}$) can be computed by point counting.

Remark 2.3. The above result applies to any finite extension K'/K as well; one just has to replace the quotient curve $\bar{Y}/I_K$ by the quotient $\bar{Y}/I_{K'}$.

Remark 2.4. For the proof of Theorem 2.1 we do not use the fact that $\bar{Y}$ is the stable reduction of Y. We can also use the special fiber of a non-minimal semistable model $\mathcal{Y}$, as long as the action of Γ_K on Y_L extends to $\mathcal{Y}$. In § 3 we encounter semistable models of Y which are minimal with respect to a certain map $\phi : Y \to X$, and for these the Γ_K-action extends.

Corollary 2.5. *Let $\bar{Y}$ be the stable reduction of Y (or the special fiber of a semistable model on which Γ_K acts, as in Remark 2.4). Let $\bar{Z} := \bar{Y}/I_K$ be the quotient curve and $\bar{Z}_{\mathbb{F}_q}$ be the natural $\mathbb{F}_q$-model of $\bar{Z}$ (see Remark 2.2). Then the local L-factor $L(Y/K, s)$ is equal to the numerator of the local zeta function of $\bar{Z}_{\mathbb{F}_q}$. More explicitly, we have*

$$L(Y/K, s) = P_1(\bar{Z}_{\mathbb{F}_q}, q^{-s}),$$

where

$$P_1(\bar{Z}_{\mathbb{F}_q}, T) := \det\left(1 - \mathrm{Frob}_q^{-1} \cdot T | H^1_{\mathrm{et}}(\bar{Z}, \mathbb{Q}_\ell)\right)$$

and $\mathrm{Frob}_q : \bar{Z}_{\mathbb{F}_q} \to \bar{Z}_{\mathbb{F}_q}$ *is the $\mathbb{F}_q$-Frobenius endomorphism.*

Proof. This is a consequence of Theorem 2.1, Remark 2.2 and the definition of $L(Y/K, s)$. □

In the remaining part of this section we give two proofs of Theorem 2.1. The first proof uses the regular model and the Picard functor. The second proof uses vanishing cycles and in particular the *Picard-Lefschetz formula*. Both proofs seem to be well known, but we do not know an explicit and convenient reference. The second proof is much closer to our view of the subject, and it gives us some additional insight that we need later on. We therefore give a rather detailed account. Along the way, we will introduce a lot of notation that will be used throughout the paper.

2.4. The first proof of Theorem 2.1 relies on the following proposition.

Proposition 2.6. *Let K be a henselian local field and Y a smooth projective curve over K. Let $\mathcal{Y}$ be an $\mathcal{O}_K$-model of Y which is semistable or regular. If $\mathcal{Y}$ is regular we assume moreover that the gcd of the multiplicities of the components of the special fiber is one. Let $\bar{Y}$ denote the* geometric *special fiber of $\mathcal{Y}$. Then the cospecialization map induces an isomorphism*

$$H^1_{\mathrm{et}}(Y_{\bar{K}}, \mathbb{Q}_\ell)^{I_K} \cong H^1_{\mathrm{et}}(\bar{Y}, \mathbb{Q}_\ell).$$

Proof. The proof is classical. By [14], Corollary 4.18, we have isomorphisms

$$H^1_{\mathrm{et}}(Y_{\bar{K}}, \mathbb{Q}_\ell(1)) \cong V_\ell(\mathrm{Pic}^0(Y)), \quad H^1_{\mathrm{et}}(\bar{Y}, \mathbb{Q}_\ell(1)) \cong V_\ell(\mathrm{Pic}^0(\bar{Y})) \tag{2}$$

and where $V_\ell(\,\cdot\,)$ denotes the rational ℓ-adic Tate module.

Let $\mathcal{J}$ denote the Néron model of the Jacobian of Y and $\mathcal{J}_s^0$ the connected component of its special fiber. Then by [9], 6.4 (see also [18], Lemma 2) we have

$$V_\ell(\mathrm{Pic}^0(Y))^{I_K} \cong V_\ell(\mathcal{J}_s^0). \tag{3}$$

On the other hand, under the conditions imposed on $\mathcal{Y}$ we have an isomorphism

$$\mathcal{J}_s^0 \cong \mathrm{Pic}^0(\bar{Y}) \tag{4}$$

by [3], Theorem 9.5.4 and Corollary 9.7.2. The proposition follows by combining (2), (3) and (4). □

The proof of Theorem 2.1 now goes as follows. Let L/K^{nr} be a finite Galois extension over which Y has semistable reduction. Let $\mathcal{Y}$ be a Γ_K-invariant semistable model of Y_L and $\bar{Y}$ its special fiber. By Proposition 2.6 we have a canonical (and therefore Γ_K-invariant) isomorphism

$$H^1_{\mathrm{et}}(Y_{\bar{K}}, \mathbb{Q}_\ell)^{I_L} \cong H^1_{\mathrm{et}}(\bar{Y}, \mathbb{Q}_\ell).$$

Taking I_K-invariants and using the Hochschild-Serre spectral sequence ([14], III.2.20), we conclude that

$$H^1_{\mathrm{et}}(Y_{\bar{K}}, \mathbb{Q}_\ell)^{I_K} \cong H^1_{\mathrm{et}}(\bar{Y}, \mathbb{Q}_\ell)^{I_K} \cong H^1_{\mathrm{et}}(\bar{Y}/I_K, \mathbb{Q}_\ell).$$

□

2.5. For our second proof of Theorem 2.1 we also prove Proposition 2.6, but only in the semistable case. Again, we let L/K^{nr} be a finite Galois extension over which Y has semistable reduction, $\mathcal{Y}$ a Γ_K-invariant semistable model of Y_L and $\bar{Y}$ the special fiber of $\mathcal{Y}$. We shall analyze $H^1_{\mathrm{et}}(Y_{\bar{K}}, \mathbb{Q}_\ell)$ via the sheaf of vanishing cycles on $\bar{Y}$.

We start with the definition of the *dual graph* of the semistable curve $\bar{Y}$, and its (co)homology. In our setup the dual graph is the undirected multigraph $\Delta_Y = (V, E)$ defined as follows. The vertex set V is in bijection with the set of irreducible components of $\bar{Y}$. The edge set E is in bijection with the singular points of $\bar{Y}$. We let $Y_{s,v}$ denote the irreducible component corresponding to $v \in V$ and $y_e \in \bar{Y}$ the singular point corresponding to $e \in E$. The edge e connects the two vertices corresponding to the components which meet in y_e. It is clear that the action of Γ_K on $\bar{Y}$ induces an action on Δ_Y.

Let $\pi : \tilde{Y}_s \to \bar{Y}$ be the normalization of $\bar{Y}$. Clearly, π is a finite map, and is an isomorphism over the smooth locus of $\bar{Y}$. Also, $\tilde{Y}_s$ is smooth, and its connected components are in bijection with the irreducible components of $\bar{Y}$ and thus with the vertex set V. Since $\bar{Y}$ is semistable, the fiber $\pi^{-1}(y_e)$, for $e \in E$, consists of exactly two distinct points. An element of $\pi^{-1}(y_e)$ is called a *branch* of $\bar{Y}$ through y_e. It will be convenient for us to choose, for all $e \in E$, one branch y_e' through y_e. The other branch through y_e is then called y_e''. We let $s(e) \in V$ (resp. $t(e) \in V$) denote the vertex of G corresponding to the component on which y_e' (resp. y_e'') lies. Thus the choice

of y_e' induces an orientation on the graph Δ_Y, which is encoded by the two maps $s,t : E \to V$ (*source* and *target*). One should keep in mind that this orientation may not be fixed by the action of Γ_K.

Let Λ be a commutative ring. Let $(C_\bullet(\Delta_Y,\Lambda),\delta)$ denote the chain complex of Λ-modules, concentrated in degree 0 and 1, where $C_0(\Delta_Y,\Lambda)$ (resp. $C_1(\Delta_Y,\Lambda)$) is the free Λ-module generated by V (resp. by E), and where $\delta : C_1(\Delta_Y,\lambda) \to C_0(\Delta_Y,\lambda)$ is defined by

$$\delta(e) := t(e) - s(e).$$

Let $(C^\bullet(\Delta_Y,\Lambda),d)$ denote the dual cochain complex. Explicitly, we have

$$C^0(\Delta_Y,\Lambda) = \Lambda^V, \quad C^1(\Delta_Y,\Lambda^E) = \Lambda^E.$$

The boundary morphism $d : C^0(\Delta_Y,\Lambda) \to C^1(\Delta_Y,\lambda)$ sends $\phi \in C^0(\Delta_Y,\Lambda)$ to the map

$$d\phi : e \mapsto \phi(s(e)) - \phi(t(e)).$$

We set

$$H_i(\Delta_Y,\Lambda) := H_i(C_\bullet(\Delta_Y,\Lambda),\delta), \quad H^i(\Delta_Y,\Lambda) := H^i(C^\bullet(\Delta_Y,\Lambda),d).$$

Clearly, these groups are zero unless $i = 0,1$, and the interesting pieces are

$$H_1(\Delta_Y,\lambda) = \ker(\delta), \quad H^1(\Delta_Y,\lambda) = \operatorname{coker}(d).$$

It is crucial for us that these constructions do not depend on the chosen orientation and that in particular, Γ_K acts on $H_1(\Delta_Y,\Lambda)$ and $H^1(\Delta_Y,\Lambda)$. (The point is that the element $-e \in C_1(\Delta_Y,\Lambda)$ represents the edge $e \in E$ with 'reversed orientation'.) Another rather obvious but important point is that $H^1(\Delta_Y,\Lambda)$ and $H_1(\Delta_Y,\Lambda)$ are free Λ-modules of finite rank, and that their definition commutes with base change with respect to an arbitrary ring morphism $\Lambda \to \Lambda'$.

By construction, we have a natural perfect pairing

$$\cup : H_1(\Delta_Y,\Lambda) \times H^1(\Delta_Y,\Lambda) \to \Lambda. \tag{5}$$

We let $\delta_e \in H^1(\Delta_Y,\Lambda)$ denote the element corresponding to $\phi_e \in C^1(\Delta_Y,\Lambda)$, with

$$\phi_e(e') = \begin{cases} 1, & e = e' \\ 0, & e \neq e' \end{cases}.$$

Note that the elements δ_e generate $H^1(\Delta_Y,\Lambda)$ and that they are 'canonical up to sign'.

Lemma 2.7. *Assume that Λ is a ring of characteristic zero. Let $(k_e)_{e\in E}$ be a collection of positive integers. Then the map*

$$\kappa : H_1(\Delta_Y,\Lambda) \to H^1(\Delta_Y,\Lambda), \quad a \mapsto \sum_e k_e(a \cup \delta_e)\delta_e$$

is injective.

Proof. The ring Λ contains the ring $\mathbb{Z}$, by assumption. Since H_1 and H^1 are free and their definition commutes with base change, it suffices to prove the lemma in the case that $\Lambda = \mathbb{Z}$. Suppose $\kappa(a) = 0$ for $a \in H_1(\Delta_Y, \mathbb{Z})$. Then

$$0 = a \cup \kappa(a) = \sum_e k_e (a \cup \delta_e)^2.$$

Since the k_e are assumed to be positive, this implies $a \cup \delta_e = 0$ for all $e \in E$. Since the δ_e generate $H^1(\Delta_Y, \mathbb{Z})$ and $\cup$ is nondegenerate, we conclude $a = 0$. □

2.6. From now on we assume that $\Lambda = \mathbb{Z}/\ell^r$ is a finite quotient of $\mathbb{Z}_\ell$. We let $\Lambda(1)$ denote the group of ℓ^rth roots on unity in K^{nr}, considered as a Λ-module (which is free of rank one). For any Λ-module M we set $M(1) := M \otimes_\Lambda \Lambda(1)$ and $M(-1) = \mathrm{Hom}(\Lambda(1), M)$.

Let $\mathcal{F}$ be defined by the following short exact sequence of sheaves of Λ-modules on $(\bar{Y})_{\mathrm{et}}$:

$$0 \to \underline{\Lambda} \to \pi_* \underline{\Lambda} \to \mathcal{F} \to 0, \tag{6}$$

where $\pi : \tilde{Y}_s \to \bar{Y}$ is the normalization of $\bar{Y}$, as above. By [14], Corollary 3.5 (c), we have

$$(\pi_* \underline{\Lambda})_y = \Lambda^{\pi^{-1}(y)}, \tag{7}$$

for a closed point $y \in \bar{Y}$. So it follows from the above description of π that $\mathcal{F}$ is a torsion sheaf, concentrated in the singular points, and that we have an isomorphism

$$\mathcal{F}_{y_e} \cong \Lambda, \tag{8}$$

for all $e \in E$. In fact, we have a canonical choice for the isomorphism (8), induced by the map

$$(\pi_* \underline{\Lambda})_y = \Lambda^{\pi^{-1}(y)} \to \Lambda, \quad \phi \mapsto \phi(y_e'') - \phi(y_e').$$

Using the projection formula and the above description of $\mathcal{F}$, the long exact sequence induced by (6) may be written as follows:

(9)

$$0 \to H^0_{\mathrm{et}}(\bar{Y}, \Lambda) \to H^0_{\mathrm{et}}(\tilde{Y}_s, \Lambda) \to H^0_{\mathrm{et}}(\bar{Y}, \mathcal{F}) \to H^1_{\mathrm{et}}(\bar{Y}, \Lambda) \to H^1_{\mathrm{et}}(\tilde{Y}_s, \Lambda) \to 0.$$

It is now easy to check that the following diagram commutes:

$$\begin{array}{ccc} H^0_{\mathrm{et}}(\tilde{Y}_s, \Lambda) & \longrightarrow & H^0_{\mathrm{et}}(\bar{Y}, \mathcal{F}) \\ \cong \downarrow & & \downarrow \cong \\ C^0(\Delta_Y, \Lambda) & \xrightarrow{d} & C^1(\Delta_Y, \Lambda). \end{array} \tag{10}$$

Here the vertical arrow on the right is induced by (8). Combining (9) and (10) we obtain a short exact sequence

$$0 \to H^1(\Delta_Y, \Lambda) \to H^1_{\mathrm{et}}(\bar{Y}, \Lambda) \to H^1_{\mathrm{et}}(\tilde{Y}_s, \Lambda) \to 0. \tag{11}$$

Clearly, all maps in (11) are Γ_K-equivariant and commute with base change in Λ.

Henceforth we consider $H^1(\Delta_Y,\Lambda)$ as a submodule of $H^1_{\rm et}(\bar{Y},\Lambda)$ via the first map in (11) and the elements $\delta_e\in H^1(\Delta_Y,\Lambda)$ defined in §2.5 as elements of $H^1_{\rm et}(\bar{Y},\Lambda)$ (the *vanishing cycles*). There is an alternative definition of these elements, as follows. Using [14], Propositions III.1.25 and III.1.27, one constructs an isomorphism

$$H^1_{\{y_e\}}(\bar{Y},\Lambda)\cong\Lambda,$$

whose sign is determined by our choice of orientation. It is then easy to see that δ_e is equal to the image of 1 under the natural map

$$\Lambda\cong H^1_{\{y_e\}}(\bar{Y},\Lambda)\to H^1(\bar{Y},\Lambda).$$

2.7. Recall that $\mathcal{Y}$ denotes the stable model of Y over $\mathfrak{o}_L$ and that $\bar{Y}=\mathcal{Y}\otimes k$ is its special fiber. Let $Y_{\bar{L}}:=Y\otimes\bar{L}$ denote the geometric generic fiber. We consider the following cartesian, Γ_K-equivariant diagram:

$$\begin{array}{ccccc}
\bar{Y} & \xrightarrow{\ i\ } & \mathcal{Y} & \xleftarrow{\ j\ } & Y_{\bar{L}} \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec} k & \xrightarrow{\ s\ } & \operatorname{Spec}\mathfrak{o}_L & \xleftarrow{\ \bar{\eta}\ } & \operatorname{Spec}\bar{L}
\end{array}\quad .$$

For $n\geq 0$, the nth sheaf of *vanishing cycles* on $\bar{Y}$ is defined as

$$\Psi^n(\underline{\Lambda}):=i^*R^nj_*(\underline{\Lambda}),$$

see [9], Exposé I, §2. For all n, $\Psi^n(\underline{\Lambda})$ is a constructible sheaf of Λ-modules on $(\bar{Y})_{\rm et}$ with a natural action of Γ_K. We have $\Psi^n(\underline{\Lambda})=0$ for $n\neq 0,1$ and

$$\Psi^0(\underline{\Lambda})=\underline{\Lambda}.$$

For $n=1$, $\Psi^1(\underline{\Lambda})$ is a torsion sheaf with support in the singular points y_e. For all $e\in E$ we have an isomorphism

$$\Psi^1(\underline{\Lambda})_{y_e}\cong\Lambda(-1),$$

canonical up to sign ([4], XV.2.2.5). From the Leray spectral sequence

$$E_2^{p,q}=H^p_{\rm et}(\bar{Y},\Psi^q(\Lambda))\Rightarrow H^{p+q}_{\rm et}(Y_{\bar{\eta}},\Lambda)$$

([9], I.2.2.3) we therefore obtain an exact and Γ_K-equivariant sequence

$$0\to H^1_{\rm et}(\bar{Y},\Lambda)\to H^1_{\rm et}(Y_{\bar{L}},\Lambda)\to\bigoplus_{e\in E}\Lambda(-1)\to H^2_{\rm et}(\bar{Y},\Lambda). \tag{12}$$

It is shown in [4], Exposé XV, that the middle map in (12) is

$$H^1_{\rm et}(Y_{\bar{L}},\Lambda)\to\bigoplus_{e\in E}\Lambda(-1),\quad a\mapsto(a\cup\delta_e)_e, \tag{13}$$

where

$$\cup: H^1_{\rm et}(Y_{\bar{L}},\Lambda)\times H^1_{\rm et}(Y_{\bar{L}},\Lambda)\to H^2_{\rm et}(Y_{\bar{L}},\Lambda)=\Lambda(-1)$$

is the cup product. (Here we use the definition of δ_e given at the end of §2.6.)

Set

$$\mathcal{K} := \ker\big(\bigoplus_{e\in E} \Lambda(-1) \to H^2_{\rm et}(\bar{Y}, \Lambda)\big).$$

Then (12) can be shortened to the short exact sequence

$$0 \to H^1_{\rm et}(\bar{Y}, \Lambda) \to H^1_{\rm et}(Y_{\bar{L}}, \Lambda) \to \mathcal{K} \to 0. \tag{14}$$

It follows from (13) that for $a \in H^1(Y_{\bar{L}}, \Lambda)$ the cup product $a \cup \delta_e$ only depends on the image of a in $\mathcal{K}$. Therefore, the cup product induces a pairing

$$H^1(\Delta_Y, \Lambda) \times \mathcal{K} \to \Lambda(-1). \tag{15}$$

This pairing is perfect. Indeed, it is nondegenerate on the left because the cup product on $H^1(Y_{\bar{L}}, \Lambda)$ is perfect (Poincaré duality) and it is nondegenerate on the right because of (13). As a consequence we obtain a canonical isomorphism

$$\mathcal{K} \cong H_1(\Delta_Y, \Lambda)(-1) \tag{16}$$

with respect to which the pairing (15) is induced by (5).

Let $\pi \in \mathcal{O}_L$ be a uniformizer and let

$$\epsilon : I_L \to \Lambda(1)$$

denote the corresponding Kummer character. For each $e \in E$ we can write the complete local ring of the stable model $\mathcal{Y}$ at the singular point y_e as $\mathcal{O}_K[[u, v \mid uv = \pi^{k_e}]]$, for a unique positive integer k_e. Using this notation, the famous *Picard-Lefschetz formula* can be stated as follows.

Proposition 2.8. *For $\sigma \in I_L$ and $a \in H^1_{\rm et}(Y_{\bar{L}}, \Lambda)$ we have*

$$\sigma(a) = a + \mathrm{Var}_\sigma(a),$$

where $\mathrm{Var}_\sigma : H^1_{\rm et}(Y_{\bar{L}}, \Lambda) \to H^1_{\rm et}(Y_{\bar{L}}, \Lambda)$ is the so called variation map *given by*

$$\mathrm{Var}_\sigma(a) := -\sum_{e\in E} k_e \epsilon(\sigma)(a \cup \delta_e)\delta_e.$$

(Note that $\epsilon(\sigma) \in \Lambda(1)$ and $a \cup \delta_e \in \Lambda(-1)$. Hence the product $k_e\epsilon(\sigma)(a \cup a)$ is naturally an element of Λ.)

Proof. See [4], Exposé XV, §3.3.-3.4. □

By the discussion above we can identify the variation map with the morphism

$$\mathcal{K} \cong H_1(\Delta_Y, \Lambda)(-1) \to H^1(\Delta_Y, \Lambda), \quad a \mapsto -\sum_{e\in E} k_e(\epsilon(\sigma)a \cup \delta_e)\delta_e,$$

where $\cup$ is the pairing (5).

2.8. We can now finish our second proof of Theorem 2.1. By the Picard-Lefschetz formula (Proposition 2.8) we have

$$H^1_{\rm et}(Y_{\bar L},\Lambda)^{I_L} = \bigcap_{\sigma\in I_L} \ker(\mathrm{Var}_\sigma). \tag{17}$$

In particular, $H^1_{\rm et}(Y_{\bar L},\Lambda)^I$ contains $H^1_{\rm et}(\bar Y,\Lambda)$. Moreover, since the fundamental character is surjective, the quotient $H^1_{\rm et}(Y_{\bar L},\Lambda)^I/H^1_{\rm et}(\bar Y,\Lambda)$ is equal to the kernel of the map

$$\mathcal{K}\cong H_1(\Delta_Y,\Lambda)(-1)\to H^1(\Delta_Y,\Lambda)(-1),\quad a\mapsto \sum_e k_e(a\cup\delta_e)\delta_e. \tag{18}$$

Both sides of (18) are free Λ-modules of finite rank, and the definition of the map commutes with arbitrary base change in Λ. Passing to the limit $\mathbb{Z}_\ell = \varprojlim_r \mathbb{Z}/\ell^r$ and choosing a generator of $\mathbb{Z}_\ell(1)$, we conclude that $H^1_{\rm et}(Y_{\bar L},\mathbb{Z}_\ell)^{I_L}$ contains $H^1_{\rm et}(\bar Y,\mathbb{Z}_\ell)$ and that the quotient can be identified (noncanonically) with the kernel of the map

$$H_1(\Delta_Y,\mathbb{Z}_\ell)\to H^1(\Delta_Y,\mathbb{Z}_\ell),\quad a\mapsto \sum_e k_e(a\cup\delta_e)\delta_e.$$

But this map is injective by Lemma 2.7 and hence

$$H^1_{\rm et}(Y_{\bar L},\mathbb{Z}_\ell)^{I_L} = H^1_{\rm et}(\bar Y,\mathbb{Z}_\ell).$$

After passing to $\mathbb{Q}_\ell$-coefficients, the rest of the proof goes as in §2.4. □

3. Admissible covers

3.1. Let $K/\mathbb{Q}_p$ be a p-adic number field as before, $X:=\mathbb{P}^1_K$ the projective line over K and $\phi: Y\to X$ a finite cover. We assume that Y is smooth, absolutely irreducible and of genus $g\geq 2$.

Let $L/K^{\rm nr}$ be a finite extension over which Y acquires semistable reduction, and let $\mathcal{Y}$ be the stable model of Y_L over $\mathcal{O}_L$. By [13], there exists a unique semistable model $\mathcal{X}$ of X_L such that ϕ extends to a finite $\mathcal{O}_L$-morphism $\mathcal{Y}\to\mathcal{X}$. Moreover, $\mathcal{Y}$ is the normalization of $\mathcal{X}$ inside the function field of Y_L. For instance, if ϕ is a Galois cover with Galois group G, then the G-action on Y_L extends to $\mathcal{Y}$ and the quotient scheme $\mathcal{X}:=\mathcal{Y}/G$ has the desired property.

Our strategy for computing the stable reduction of Y is to try to reverse the process described above: try to find a semistable model $\mathcal{X}$ of X whose normalization $\mathcal{Y}$ with respect to Y is again semistable. Let us call such a model *ϕ-semistable*. We have seen above that ϕ-semistable models exist (and there is a unique minimal one). But that does not mean that finding one is easy.

Suppose that ϕ is a Galois cover with Galois group G. Suppose, moreover, that p does not divide the order of G. In this case it is well known how to determine a ϕ-semistable model. The main insight goes back to [11] and is based on the notion of *admissible covers*. The purpose of the present section is to recall this result.

In [2] a general method for finding a ϕ-semistable model $\mathcal{X}$ is developed. In the special case where ϕ is a Galois cover and the group G is cyclic of order p (where p is, as before, the residue characteristic of K) this approach has been made algorithmic and practical in [1]. In general, however, we are still lacking a practical procedure to find $\mathcal{X}$.

3.2. We first need a generalization of the notion of a (semi)stable model.

Definition 3.1. Let S be a scheme and $\mathcal{X} \to S$ a semistable curve over S. Let $\mathcal{X}^{\rm sm} \subset \mathcal{X}$ denote the smooth locus of $\mathcal{X} \to S$. A *marking* on $\mathcal{X}/S$ of degree d is a closed subscheme $\mathcal{D} \subset \mathcal{X}^{\rm sm}$ such that $\mathcal{D} \to S$ is finite étale of degree d. The pair $(\mathcal{X}/S, \mathcal{D})$ is called a *semistably marked curve*.

The semistable reduction theorem easily extends to the marked case, as follows (see [12]). Let K be as before and X/K a smooth projective curve, which is absolutely irreducible of genus g. Let $D \subset X$ be a marking of degree d. Then there exists a finite extension $L/K^{\rm nr}$ and a semistable model $\mathcal{X}$ of $X_L := X \otimes_K L$ over $\mathcal{O}_L$ such that the Zariski closure $\mathcal{D} \subset \mathcal{X}$ of $D_L \subset X_L$ is a marking. Moreover, if $2g - 2 + d > 0$ then there exists a unique minimal semistable model $\mathcal{X}$ with this property (the *stably marked model*).

Remark 3.2.

(1) Following our convention from § 2, we consider finite extensions L of the maximal unramified extension of K. The residue field k of L is therefore separably closed. Let $(\mathcal{X}, \mathcal{D})$ be a semistable marked model of (X_L, D_L), and let $(\bar{X}, D_s)$ denote its special fiber. Then $D_s = \{\bar{x}_1, \ldots, \bar{x}_d\}$ consists of d pairwise distinct *smooth* points of $\bar{X}$. As $\mathcal{O}_L$ is henselian, this implies that D_L is *split*, i.e. $D_L = \{x_1, \ldots, x_d\}$ consists of d distinct L-rational points. So a necessary condition for (X, D) to have stable reduction over L is that D splits over L.
(2) Now suppose that $X = \mathbb{P}^1_K$. Then the necessary condition from (1) is also sufficient: (X, D) has semistable reduction over any finite extension $L/K^{\rm nr}$ which splits D. See e.g. [8].

3.3. We now return to the situation from the beginning of this section. Let $\phi : Y \to X := \mathbb{P}^1_K$ be a finite cover of the projective line, where Y is smooth and absolutely irreducible over K. We make the following additional assumptions on ϕ:

(a) The cover ϕ is *potentially Galois*, i.e. the base change $\phi_{\bar{K}} : Y_{\bar{K}} \to X_{\bar{K}}$ is a Galois cover.
(b) Let G denote the Galois group of $\phi_{\bar{K}}$. Then the residue characteristic p of K does not divide the order of G.

Let $D \subset X$ be the *branch locus* of ϕ, i.e. the reduced closed subscheme supporting exactly the branch points of ϕ. Then $D \to \operatorname{Spec} K$ is a finite flat morphism. Since the characteristic of K is assumed to be zero and D is reduced by definition, $D \to \operatorname{Spec} K$ is actually étale and hence D is a marking on X. The geometric points of D are exactly the branch points of

$\phi_{\bar{K}}$. Let d denote the degree of D, i.e. the number of branch points of $\phi_{\bar{K}}$. The assumption $g \geq 2$ implies that $d \geq 3$.

Let L/K^{nr} be a finite extension which splits D. By Remark 3.2 (2) the pair (X, D) has semistable reduction over L. Let $(\mathcal{X}, \mathcal{D})$ denote the stably marked model of (X_L, D_L). Let $\mathcal{Y}$ denote the normalization of $\mathcal{X}$ in the function field of Y. Then $\mathcal{Y}$ is a normal integral model of Y over $\mathcal{O}_L$, but in general $\mathcal{Y}$ is not semistable. Let $\bar{Y} := \mathcal{Y} \otimes k$ be the special fiber and $\phi_s : \bar{Y} \to \bar{X}$ the induced map.

An irreducible component W of $\bar{Y}$ corresponds to a discrete valuation η_W of the function field of Y_L (since W is a prime divisor on $\mathcal{Y}$). Let m_W denote the ramification index of η_W in the extension of function fields induced by ϕ. The integer m_W is called the *multiplicity* of the component W. (Alternatively, one can define m_W as the length of $\mathcal{O}_{\mathcal{Y},W}/(\pi)$, where $\mathcal{O}_{\mathcal{Y},W}$ is the local ring at the generic point of W and π is a prime element of $\mathcal{O}_L$.)

Theorem 3.3. *Assume that*

(a) $\phi_L : Y_L \to X_L$ *is a Galois cover, and*
(b) $m_W = 1$ *for every irreducible component* W *of* $\bar{Y}$.

Then $\mathcal{Y}$ *is a semistable model of* Y_L.

Proof. (cf. [13], Theorem 2.3) Let $\mathcal{U} := \mathcal{X}^{\mathrm{sm}}$ denote the smooth locus and $\mathcal{V} \subset \mathcal{Y}$ the inverse image of $\mathcal{U}$. Since $\mathcal{U}$ is regular, the branch locus of the cover $\mathcal{V} \to \mathcal{U}$ is a divisor (by Purity). Hence the assumption $m_W = 1$ for all W implies that $\mathcal{V} \to \mathcal{U}$ is unramified along the special fiber. Therefore the branch locus of $\mathcal{V} \to \mathcal{U}$ is equal to the marking $\mathcal{D} \subset \mathcal{U}$ (i.e. the Zariski closure of the branch locus of $\phi_L : Y_L \to X_L$). Since $\mathcal{D} \to \operatorname{Spec} \mathcal{O}_L$ is smooth and p does not divide the order of the Galois group ϕ_L, $\mathcal{V} \to \mathcal{U}$ is tamely ramified along $\mathcal{D}$ (see [10], § 2). Now the smoothness of $\mathcal{U} \to \operatorname{Spec} \mathcal{O}_L$ implies the smoothness of $\mathcal{V} \to \operatorname{Spec} \mathcal{O}_L$.

It remains to be shown that every point $y \in \mathcal{Y} \backslash \mathcal{V}$ is an ordinary double point of $\bar{Y}$. By construction, the image point $x = \phi_s(y)$ is an ordinary double point. Let $A := \hat{\mathcal{O}}_{\mathcal{X},x}$ and $B := \hat{\mathcal{O}}_{\mathcal{Y},y}$ denote the complete local rings at x and y, respectively. All rings involved are excellent and hence completion commutes with normalization. It follows that B is the integral closure of A in a finite Galois extension of the fraction field of A. Moreover, the Galois group of B/A is a subgroup of the Galois group of ϕ_L and is therefore of order prime to p. The theorem follows now from the following crucial lemma. □

Lemma 3.4. *Let* $(R, \mathfrak{m})$ *be a complete local ring with separably closed residue field,* $a \in \mathfrak{m} \backslash \{0\}$ *and* $A := R[[u, v \mid uv = a]]$. *Let* B *be the integral closure of* A *in a finite Galois extension of* $\mathrm{Frac}(A)$. *Assume that the map* $\operatorname{Spec} B \to \operatorname{Spec} A$ *is étale away from the closed point of* $\operatorname{Spec} A$. *Then the following holds.*

(1) *The Galois group of B/A is cyclic, and its order n is invertible in R.*
(2) *There exist elements $b \in R$ and $s, t \in B$ such that*

$$b = st, \quad s^n = u, \quad t^n = v.$$

Furthermore, we have $B = R[[s, t \mid st = b]]$.

Proof. See [15], Théorème 3.2 and Remarque 3.3. □

Corollary 3.5. *Let $\phi : Y \to X = \mathbb{P}^1_K$ be a cover as before, with branch locus $D \subset X$. Assume that the order of the Galois group of $\phi_{\bar{K}}$ is prime to the residue characteristic p of K. Let $L_0/K^{\rm nr}$ be a finite extension which splits D and such that ϕ_{L_0} is Galois. Then there exists a tamely ramified extension L/L_0 over which Y has semistable reduction.*

Proof. Let $(\mathcal{X}_0, \mathcal{D}_0)$ be the stably marked model of (X, D) over $\mathcal{O}_{L_0}$ and $\mathcal{Y}_0$ the normalization of $\mathcal{X}_0$ in Y_{L_0}. Let e be the lcm of all multiplicities m_W, where W runs over the irreducible components of the special fiber of $\mathcal{Y}_0$. It is clear that e divides the order of the Galois group of ϕ_{L_0} and is therefore prime to p.

Let L/L_0 be the unique tame extension of degree e. Let $(\mathcal{X}, \mathcal{D})$ be the base change of $(\mathcal{X}_0, \mathcal{D}_0)$ to $\mathcal{O}_L$; this is still a stably marked curve. Let $\mathcal{Y}$ be the normalization of $\mathcal{X}$ in Y_L. It follows from Abhyankar's lemma that the multiplicities of the irreducible components of $\bar{Y}$ are one. By Theorem 3.3, $\mathcal{Y}$ is semistable. This proves the corollary. □

3.4. Let $\phi : Y \to X = \mathbb{P}^1_K$ be a finite cover of order prime to p which is potentially Galois. Let us briefly summarize what we have learned about the problem of computing the semistable reduction of Y so far.

Let $L_0/K^{\rm nr}$ be a field extension that makes the branch divisor D of ϕ rational and such that ϕ_{L_0} is Galois. Let L/L_0 be the (unique) tame extension of degree $n = \deg(\phi)$. Let $(\mathcal{X}, \mathcal{D})$ be the stably marked model of (X, D) (which exists by Remark 3.2 (2)). Then the normalization $\mathcal{Y}$ of $\mathcal{X}$ in Y_L is a semistable model of Y. In the following section we will make this abstract procedure totally explicit in the case where ϕ is a Kummer cover. We end this section with two remarks on the general case.

Remark 3.6. (1) The semistable model $\mathcal{Y}$ constructed above is in general not the stable model. Furthermore, the extension $L/K^{\rm nr}$ is in general not the minimal extension over which Y has semistable reduction.
(2) The proof of Theorem 3.3 shows more than just the semistability of $\mathcal{Y}$; it shows precisely that $\mathcal{Y} \to \mathcal{X}$ is an *admissible cover* (see [11] or [20]). For the purpose of the present paper, we do not need to know what that means. However, the following consequence will be useful. Smooth (resp. singular) point of $\bar{Y}$ map to smooth (resp. singular) points of $\bar{X}$. Since the irreducible components of $\bar{X}$ are smooth (see 4.2 below), it follows that the same holds for the irreducible components of $\bar{Y}$.

4. Kummer covers

4.1. As before, $K/\mathbb{Q}_p$ is a finite extension. Let $\phi : Y \to X := \mathbb{P}^1_K$ be the cover of curves given by the Kummer equation

$$y^n = f(x),$$

where $f(x)$ is a polynomial in the parameter x of X and n is prime to p. Then ϕ satisfies Conditions (a) and (b) of §3.3. In fact, the base change of ϕ to $K^{\rm nr}$ is a Galois cover with cyclic Galois group of order n.

It is no restriction to assume that $f(x)$ has the form

$$f = \prod_{i=1}^r f_i^{a_i},$$

where $f_i \in K[x]$ is nonconstant and irreducible and $0 < a_i < n$. We may also assume that $\gcd(n, a_1, \ldots, a_r) = 1$. This means that Y is absolutely irreducible.

Our goal is to compute the stable reduction of Y in terms of the data $f(x)$ and n. We will follow the procedure suggested by Theorem 3.3 and Corollary 3.5. This procedure gives us a finite extension $L/K^{\rm nr}$ and a semistable model $\mathcal{Y}$ of Y over $\mathcal{O}_L$.

4.2. Let $D \subset X$ be the (reduced) branch divisor of ϕ. The geometric points of D are precisely the roots of the polynomial f, possibly together with the point ∞. In fact, $\infty \in D$ if and only if

$$\sum_i a_i \deg(f_i) \not\equiv 0 \pmod{n}.$$

We may define $0 \leq a_0 < n$ by $\sum_{i=0}^r a_i \deg(f_i) \equiv 0 \pmod{n}$, where we put $\deg(f_0) =: 1$. Then a_0 describes the ramification at ∞.

Let L_0 be the splitting field of f over $K^{\rm nr}$. Then D splits over L_0, and we can write $D_{L_0} = \{\alpha_1, \ldots, \alpha_d\}$. Let L/L_0 be a finite extension. (We describe what extension to choose in § 4.3.) The first step towards computing the stable reduction of Y is to construct the stably marked model $(\mathcal{X}, \mathcal{D})$ of (X, D) over $\mathcal{O}_L$ (Remark 3.2 (2)).

Let us call an isomorphism $\lambda : X_L \xrightarrow{\sim} \mathbb{P}^1_L$ a *chart*. Since $X_L = \mathbb{P}^1_L$ by definition, a chart may be represented by an element in $\mathrm{PGL}_2(L)$. We call two charts λ_1, λ_2 *equivalent* if the automorphism $\lambda_2 \circ \lambda_1^{-1} : \mathbb{P}^1_L \xrightarrow{\sim} \mathbb{P}^1_L$ extends to an automorphism of $\mathbb{P}^1_{\mathcal{O}_L}$, i.e. corresponds to an element of $\mathrm{PGL}_2(\mathcal{O}_L)$. Then we have:

Proposition 4.1. *Let $\lambda_1, \ldots, \lambda_m : X_L \to \mathbb{P}^1_L$ be a finite set of charts. Then there exists a unique semistable model $\mathcal{X}$ of X_L with the following property.*

> *A chart $\lambda : X_L \xrightarrow{\sim} \mathbb{P}^1_L$ extends to a morphism $\mathcal{X} \to \mathbb{P}^1_{\mathcal{O}_L}$ if and only if λ is equivalent to one of the charts λ_i.*

Let $\lambda_{s,i} : \bar{X} \to \mathbb{P}^1_k$ be the map induced by the extension of λ_i. Then for all i there exists a unique irreducible component $X_{s,i}$ of $\bar{X}$ such that $\lambda_{s,i}$ induces

an isomorphism $X_{s,i} \stackrel{\sim}{\to} \mathbb{P}^1_k$ and contracts all other components. This gives a bijection between the equivalence classes of the charts λ_i and the irreducible components of $\bar{X}$.

Proof. See e.g. [17], Proposition 1.1. □

For every triple $(\alpha_a, \alpha_b, \alpha_c)$ of distinct elements of D_L we let $\lambda = \lambda_{(a,b,c)}$ denote the unique chart such that

$$\lambda(\alpha_a) = 0, \quad \lambda(\alpha_b) = 1, \quad \lambda(\alpha_c) = \infty.$$

Proposition 4.2. *Let $\mathcal{X}$ denote the semistable model of X_L corresponding to the set of equivalence classes of all charts $\lambda_{(a,b,c)}$ as above. Then $D_L \subset X_L$ extends to a marking $\mathcal{D} \subset \mathcal{X}$, and $(\mathcal{X}, \mathcal{D})$ is the stably marked model of (X_L, D_L).*

Proof. Let $\mathcal{D}$ denote the Zariski closure of D_L inside $\mathcal{X}$. Let $\bar{\alpha}_a \in D_s \subset \bar{X}$ denote the specialization of $\alpha_a \in D_L$. In order to prove that $\mathcal{D} \subset \mathcal{X}$ is a marking, it suffices to show that the points $\bar{\alpha}_1, \ldots, \bar{\alpha}_d$ are pairwise distinct smooth points of $\bar{X}$.

Given two distinct indices a, b, we can choose a third index c, distinct from a, b. Then the chart $\lambda = \lambda_{(a,b,c)}$ induces a morphism $\lambda_s : \bar{X} \to \mathbb{P}^1_k$ such that $\lambda_s(\bar{\alpha}_a) = 0$ and $\lambda_s(\bar{\alpha}_b) = 1$. It follows that $\bar{\alpha}_a \neq \bar{\alpha}_b$.

Let $x \in \bar{X}$ be a singular point. Since $\bar{X}$ is a 'tree of projective lines', it is the union of two connected closed subsets $\bar{X}', \bar{X}'' \subset \bar{X}$ with $\bar{X}' \cap \bar{X}'' = \{x\}$. Let S' (resp. S'') be the set of indices a such that $\bar{\alpha}_a \in \bar{X}'$ (resp. $\bar{\alpha}_a \in \bar{X}''$). Since the $\bar{\alpha}_a$ are pairwise distinct, we have $|S' \cap S''| \leq 1$. We claim that $|S'|, |S''| \geq 2$. To see this, choose (a, b, c) such that the map $\lambda_s : \bar{X} \to \mathbb{P}^1_k$ induced by $\lambda = \lambda_{(a,b,c)}$ is nonconstant on $\bar{X}'$. Then λ_s contracts $\bar{X}''$ and maps $\bar{\alpha}_a, \bar{\alpha}_b, \bar{\alpha}_c$ to pairwise distinct points of $\mathbb{P}^1_k$. It follows that at least two of the three indices a, b, c must lie in S', proving the claim.

Now assume that $\bar{\alpha}_a = x$, i.e. $a \in S' \cap S''$. Then we can choose $b \in S' \backslash \{a\}$ and $c \in S'' \backslash \{a\}$. Consider the map $\lambda_s : \bar{X} \to \mathbb{P}^1_k$ induced by $\lambda = \lambda_{(a,b,c)}$. Since $\bar{\alpha}_a, \bar{\alpha}_b, \bar{\alpha}_c$ have pairwise distinct images, λ_s is nonconstant on $\bar{X}'$ and on $\bar{X}''$. But this is impossible because we know from Proposition 4.1 that λ_s contracts all but one component of $\bar{X}$. Therefore, all points $\bar{\alpha}_a$ must be smooth points of $\bar{X}$. We conclude that $\mathcal{D}$ is a marking of the semistable curve $\mathcal{X}$.

It remains to be shown that $(\mathcal{X}, \mathcal{D})$ is stably marked. Let $\bar{Z}$ be an irreducible component of $\bar{X}$. Choose (a, b, c) such that the reduction λ_s of $\lambda = \lambda_{(a,b,c)}$ induces an isomorphism $\bar{Z} \stackrel{\sim}{\to} \mathbb{P}^1_k$. Let $x_0, x_1, x_\infty \in \bar{Z}$ denote the inverse images of $0, 1, \infty \in \mathbb{P}^1_k$. If $\bar{\alpha}_a$ lies on $\bar{Z}$ then $\bar{\alpha}_a = x_0$. Otherwise, it lies on a component of $\bar{X}$ that is contracted by λ_s to the point $0 \in \mathbb{P}^1_k$. But then $\bar{Z}$ intersects another component of $\bar{X}$ in x_0. It follows that x_0 is either equal to $\bar{\alpha}_a$ or is a singular point of $\bar{X}$. The same fact holds for x_1 and x_∞. Therefore, the marked curve $(\bar{X}, D_s)$ satisfies the 'three point criterion' for stability, and $(\mathcal{X}, \mathcal{D})$ is stably marked. □

4.3. The stably marked curve $(\bar{X}, D_s)$ constructed above is called the *stable reduction* of (X, D). Clearly, it is independent of the chosen extension $L/K^{\rm nr}$, and there is a natural semilinear action of Γ_K on $(\bar{X}, D_s)$.

We would like to have a more algorithmic construction of $(\bar{X}, D_s)$. In view of the abstract construction of $(\mathcal{X}, \mathcal{D})$ from the previous subsection, this is relatively straightforward. Our input is the set $D_L = \{\alpha_1, \ldots, \alpha_d\} \subset \mathbb{P}^1_L$. The desired output consists of

- the dual graph $\Delta_X = (V, E)$ of $\bar{X}$ (which is actually a tree),
- the *marking* of Δ_X induced by the marking $D_s \subset \bar{X}$, i.e. the map
$$\psi : S := \{1, \ldots, d\} \to V$$
which send $a \in S$ to the vertex v corresponding to the component on which $\bar{\alpha}_a$ lies, and
- a family of charts $(\lambda_v)_{v \in V}$ such that the reduction $\lambda_{s,v}$ of λ_v is an isomorphism on the component $X_{s,v}$.

It is clear the the marked curve $(\bar{X}, D_s)$ can be reconstructed from these data.

We construct $\Delta_X, \psi, (\lambda_v)_{v \in V}$ inductively. We start with a graph $\Delta^{(0)}$ with a single vertex v_0 and no edges. We choose a triple (a, b, c) of distinct elements of S and set $\lambda_{v_0} := \lambda_{(a,b,c)}$ (see the previous subsection for the notation). We now divide S into s nonempty distinct subsets, $S = S_1^{(1)} \cup \ldots \cup S_s^{(1)}$, as follows. Two indices $a', b' \in S$ lie in the same subset if and only if the images $\lambda_{v_0}(\alpha_{a'}), \lambda_{v_0}(\alpha_{b'}) \in \mathbb{P}^1_L$ have the same reduction on $\mathbb{P}^1_k$. By choice of λ_{v_0}, a, b, c lie in distinct subset, so we have $s \geq 3$.

For $i = 1, \ldots, s$ we do the following. If $S_i^{(1)} = \{a\}$ has a single element then we set $\psi(a) := v_0$. Otherwise, we add to the graph $\Delta^{(0)}$ a new vertex v_i and an edge joining v_0 and v_i. We also choose a new triple (a, b, c) with $a \notin S_i^{(1)}$ and $b, c \in S_i^{(1)}$ and set $\lambda_{v_i} := \lambda_{(a,b,c)}$. The new graph is called $\Delta^{(1)}$. We also have a partial definition of ψ, marking $\Delta^{(1)}$ with a subset of S, and a family of charts λ_v indexed by the vertices of $\Delta^{(1)}$.

It is clear how to proceed. For each of the new vertices v_i, we subdivide $S_i^{(1)}$ into subsets $S_j^{(2)}$ according to the reduction of $\lambda_{v_i}(\alpha_a)$, for $a \in S_i^{(1)}$. If $S_j^{(2)} = \{a\}$ has a single element, then we set $\psi(a) := v_i$. Otherwise, we add to $\Delta^{(1)}$ a new vertex v_j and an edge joining v_i and v_i. We choose a triple (a, b, c) such that $a \neq S_j^{(2)}$ and $b, c \in S_j^{(2)}$ and set $\lambda_j := \lambda_{(a,b,c)}$. This gives us a new graph $\Delta^{(2)}$ and partial definitions of ψ and (λ_v). We continue in the same manner, until no new vertices can be added to $\Delta^{(m)}$ anymore because none of the sets $S_j^{(m)}$ has more than one element. Then $\Delta^{(m)} = \Delta_X$, and ψ and (λ_v) are completely defined.

4.4. How can we describe the action of Γ_K on $(\bar{X}, D_s)$ in terms of the data $(\Delta_X, \psi, (\lambda_v)_{v \in V})$? It is clear that the induced action of Γ_K on the tree Δ_X

is completely determined, via the marking ψ, by the natural action of Γ_K on the set $D_L = \{\alpha_1, \ldots, \alpha_d\}$. For $v \in V$ let us denote by $\Gamma_v \subset \Gamma_K$ the stabilizer of v. In addition to the action of Γ_K on Δ_X, it will be sufficient for our purposes to know the action of Γ_v on $X_{s,v}$, for all $v \in V$. This is done by the lemma below.

Fix $v \in V$ and $\sigma \in \Gamma_v$. We identify the component $X_{s,v}$ with $\mathbb{P}^1_k$ via the reduction of the chart λ_v. We also denote by $\sigma : X_L \xrightarrow{\sim} X_L$ the automorphism induced by σ via the identification $X_L = X \otimes_K L$.

Lemma 4.3. *The automorphism $\sigma_v := \lambda_v \circ \sigma \circ \lambda_v^{-1} : \mathbb{P}^1_L \xrightarrow{\sim} \mathbb{P}^1_L$ extends to an automorphism of $\mathbb{P}^1_{\mathcal{O}_L}$. Its reduction $\bar{\sigma}_v : \mathbb{P}^1_k \xrightarrow{\sim} \mathbb{P}^1_k$ is the automorphism induced by σ via the action of Γ_v on $X_{s,v}$ and the identification $X_{s,v} \cong \mathbb{P}^1_k$.*

Proof. Set $\lambda_v' := \sigma^{-1} \circ \lambda_v \circ \sigma$. Both charts λ_v, λ_v' extend to a map $\mathcal{X} \to \mathbb{P}^1_{\mathcal{O}_L}$ whose restriction to the special fiber is an isomorphism on $X_{s,v}$ and contracts all other components. By [12], such a map is unique up to isomorphism of the target (i.e. the charts λ_v and λ_v' are equivalent, as defined in § 4.2). It follows that $\sigma_v = \sigma \circ \lambda_v' \circ \lambda_v^{-1}$ extends to an automorphism of $\mathbb{P}^1_{\mathcal{O}_L}$. This proves the first statement of the lemma. The second statement is a formal consequence of the first. □

4.5. Recall that we are looking at the Kummer cover $\phi : Y \to X = \mathbb{P}^1_K$ given by the equation

$$y^n = f(x).$$

We have constructed the stably marked model $(\mathcal{X}, \mathcal{D})$ of (X, D), where $D \subset X$ is the branch locus of ϕ. Furthermore, $\mathcal{X}$ is defined over $\mathcal{O}_{L_0}$, where L_0/K^{nr} is a finite extension which splits D. Let $\mathcal{Y}$ be the normalization of $\mathcal{X}$ in Y. By Corollary 3.5, $\mathcal{Y}$ is a semistable model of Y if we choose for L a sufficiently large tame extension of L_0, the splitting field of D. The following proposition tells us how large L has to be exactly, and describes the special fiber $\bar{Y}$ of $\mathcal{Y}$.

We first need some more notation. Fix $v \in V$. Let η_v denote the normalized discrete valuation of the function field $L(X)$ corresponding to the irreducible component $X_{s,v}$ of $\bar{X}$. Since $\bar{X}$ is reduced, the extension $L(X)/L$ is *weakly unramified* with respect to η_v (i.e. any prime element of $\mathcal{O}_L$ is a prime element for η_v). For an explicit description of η_v, we write $L(X) = L(x_v)$, where x_v is the pullback of the standard parameter of $\mathbb{P}^1_L$ via the chart $\lambda_v : X_L \xrightarrow{\sim} \mathbb{P}^1_L$. Then η_v is simply the Gauss valuation of $L(x_v)$ with respect to the parameter x_v and the standard normalized valuation on L.

We choose a prime element π of $\mathcal{O}_L$ and define

$$N_v := \eta_v(f), \quad f_v := \pi^{-N_v} f.$$

Then $\eta_v(f_v) = 0$. Let $\bar{f}_v$ denote the image of f_v in the residue field $k(\eta_v)$ of η_v. Let n_v denote the order of the image of $\bar{f}_v$ in the group $k(\eta_v)^\times/(k(\eta_v)^\times)^n$.

Proposition 4.4. (1) *The model $\mathcal{Y}$ of Y_L is semistable if and only if $n|N_v$, for all $v \in V$.*

(2) *Assume that the condition in (1) holds true, and fix $v \in V$. Then there is a bijection between the set of irreducible components of $\bar{Y}$ lying above $X_{s,v}$ and the set of elements $\bar{g} \in k(\eta_v)$ satisfying*

$$\bar{g}^{n/n_v} = \bar{f}_v.$$

If $Y_{s,v}$ is the component corresponding to $\bar{g}$, then the map $Y_{s,v} \to X_{s,v}$ is the Kummer cover with equation

$$\bar{y}_v^{n_v} = \bar{g}.$$

Proof. By Theorem 3.3 and the proof of Corollary 3.5, $\mathcal{Y}$ is semistable if and only if the valuation η_v is unramified in the extension of function fields $L(Y)/L(X)$, for all $v \in V$. Furthermore, if this is the case, then the irreducible components of $\bar{Y}$ above $X_{s,v}$ are in bijection with the discrete valuations on $L(Y)$ extending η_v. More precisely, if ξ_v is an extension of η_v to $L(Y)$, then the corresponding component is the smooth projective model of the function field of $k(\xi_v)$. This reduces the proof of the proposition to standard facts on valuations in Kummer extensions. For convenience we give the main argument.

Assume that $n|N_v$ for some v. Then the element $y_s := \pi^{-N_v/n}y \in L(Y)$ generates the extension $L(Y)/L(X)$ and is a root of the irreducible polynomial $F_v := T^n - f_v$. The polynomial F_v is integral with respect to η_v and its reduction is separable and is the product of n/n_v irreducible factors of degree n_v, as follows:

$$\bar{F}_v = \prod_{\bar{g}^{n/n_v} = \bar{f}_v} (T^{n_v} - \bar{g}).$$

It follows that η_v is unramified in the extension $L(Y)/L(X)$. Furthermore, the extensions of η_v are in bijection with the irreducible factors of $\bar{F}_v$. For each extension the residue field extension is generated by the image of y_v, which is a root of the corresponding irreducible factor of $\bar{F}_v$. This proves (2) and the forward implication in (1). The reverse implication in (1) is left to the reader. □

Using Proposition 4.4, we are able to describe the stable reduction $\bar{\phi} : \bar{Y} \to \bar{X}$ of the cover $\phi : Y \to X$, knowing the following data:

- the data $(\Delta_X, \psi, (\lambda_v)_{v \in V})$ from §4.3,
- the valuations $N_v = \eta_v(f)$, $v \in V$, and
- the rational functions $\bar{f}_v \in k(x_v)$, $v \in V$.

The following lemma shows that the data $N_v, \bar{f}_v$ is essentially determined by $(\Delta_X, \psi, (\lambda_v)_{v \in V})$ and the exponents a_i.

Remark 4.5. (1) We identify $k(\eta_v)$ with the rational function field $k(x_v)$. Then

$$\bar{f}_v = c \prod_{\substack{\psi(i)=v \\ \lambda_{s,v}(\bar{\alpha}_i) \neq \infty}} (x_v - \lambda_{s,v}(\bar{\alpha}_i))^{a_i},$$

where $c\in k^\times$ is a constant and $i\in S=\{1,\ldots,n\}$ (for the notation ψ, $\lambda_{s,v}$ and $\bar\alpha_i$ see §4.2-4.3).

(2) We have
$$n_v=\frac{n}{\gcd(n,a_i)},$$
where $\psi(i)=v$.

The cover $\bar Y\to\bar X$ can be reconstructed from $\bar Y_v\to\bar X_v$ by purely combinatorial arguments. This is because $\bar Y\to\bar X$ is an admissible G-cover. However, for our computation of local L-factors we do not need this (as we will see).

4.6. We now describe the action of Γ_K on $\bar Y$. Having described $\bar Y$ explicitly in the previous section, this poses no problem. For every component $\bar Y_v$ of the stable reduction we have determined an explicit coordinate y_v. It is obvious from the definition to see how the elements of the stabilizer Γ_v act on this component.

Mostly it is not necessary to compute the action of Γ_v on $\bar Y_v$ via the coordinate as explained above. Instead, it suffices to determine how Γ_K permutes the ramification points of ϕ. Reasonably often this already completely determines the action of Γ_K on $\bar Y$. This is for example the case in the example treated in § 6. See also Example 5.5.

5. Action of the monodromy group of the stable reduction

5.1. We start by recalling the situation. Let $\bar\phi_i:\bar Y_i\to\bar X_i$ be an irreducible component of the stable reduction of a Kummer cover as in § 4.1. We write $\tau\in G_i:=\mathrm{Gal}(\bar Y_i,\bar X_i)\simeq\mathbb{Z}/n_i\mathbb{Z}$ for a generator of the geometric Galois group. Let $I_{\bar Y_i}$ (resp. $I_{\bar X_i}$) be the (finite) image of I_K in $\mathrm{Aut}_k(\bar Y_i)$ (resp. $\mathrm{Aut}_k(\bar X_i)$). In this section, we describe the *inertia quotient*
$$\bar Y_i/I_{\bar Y_i}\to\bar X_i/I_{\bar X_i}$$
explicitly. The following easy lemma describes the structure of $I_{\bar Y_i}$.

Lemma 5.1. (a) *We have $I_{\bar Y_i}\simeq P_i\rtimes\mathbb{Z}/m_i\mathbb{Z}$, where $P_i\simeq\mathbb{F}_p^{s_i}$ and $p\nmid m_i$.*

(b) *The intersection of P_i with G_i inside* $\mathrm{Aut}_k(\bar Y_i)$ *is contained in the center of $\langle I_{\bar Y_i},G_i\rangle$.*

(c) *The groups $I_{\bar Y_i}$ and G_i commute.*

Proof. Statement (c) follows from the assumption that the G-Galois action is defined over K^{nr}. The first part of (a) immediately follows from the definition of $I_{\bar Y_i}$ as quotient of the inertia group I_K. Let P_i be the Sylow p-subgroup of $I_{\bar Y_i}$. Since G_i and $I_{\bar Y_i}$ commute and $p\nmid|G_i|$, we may regard P_i as subgroup of $\langle I_{\bar Y_i},G_i\rangle/G_i\subset\mathrm{Aut}_k(\bar X_i)$. Since the genus of $\bar X_i$ is zero, it follows that P_i is an elementary abelian p-group. This proves (a). Part (b) is well-known. $\square$

We recall from Section 4.2 that we haven chosen parameters $\bar{y}_i, \bar{x}_i$ of $\bar{Y}_i, \bar{X}_i$, respectively, such that ϕ_i is given by

$$\bar{y}_i^n = \bar{f}_i(\bar{x}_i). \tag{19}$$

Moreover, we know how the monodromy group acts on these parameters (§ 4.6). We compute the inertia quotient in three steps.

(I) We first divide out by $G_i \cap I_{\bar{Y}_i}$. Since this group is a subgroup of the geometric Galois group G_i, the quotient is still a Kummer cover, hence easy to describe.

(II) Then we divide out by $P_i \subset I_{\bar{Y}_i}/G_i \cap I_{\bar{Y}_i}$. Since P_i is an elementary abelian p-group, the invariants are additive polynomials.

(III) We are now reduced to the case that $G_i \cap I_{\bar{Y}_i} = P_i = \{1\}$. This case can be dealt with explicitly.

Step I: Dividing out by $G_i \cap I_{\bar{Y}_i}$. The group $G_i \cap I_{\bar{Y}_i}$ obviously acts trivially on $\bar{X}_i$. Denote $\bar{n}_i = |G_i/G_i \cap I_{\bar{Y}_i}|$. The quotient $\bar{Y}_i/G_i \cap I_{\bar{Y}_i}$ is given by the equation

$$\bar{z}_i^{\bar{n}_i} = \bar{f}_i(\bar{x}_i), \qquad \text{where } \bar{z}_i = \bar{y}_i^{n_i/\bar{n}_i}.$$

We see that computing the invariants by $G_i \cap I_{\bar{Y}_i}$ is easy. We only have to determine the order of $G_i \cap I_{\bar{Y}_i}$. The following lemma gives an upper bound for this order.

Lemma 5.2. *Let $\sigma \in I_{\bar{Y}_i}$ be an inertia element.*

(a) *Then*

$$\sigma^* \bar{f}_i = c_i \frac{\bar{f}_i}{g(\bar{x}_i)^{n_i}},$$

with $\bar{f}_i, g \in k(\bar{x}_i)$ and $c_i \in k$ a root of unity.

(b) *Let m be the order of $\sigma \in \mathrm{Aut}(\bar{X}_i)$. Then the order of $\sigma \in Aut(\bar{Y}_i)$ divides $\mathrm{lcm}(m, n_i)$. In particular, $|G_i \cap I_{\bar{Y}_i}|$ divides $\gcd(m, n_i)$.*

Proof. Let σ be as in the statement of the lemma. Kummer theory, together with Lemma 5.1.(c), implies that $\sigma^* \bar{f}_i = c_i \bar{f}_i / g(\bar{x}_i)^{n_i}$. Since σ has finite order, it follows that c_i is a root of unity. Note that the order of c_i divides the order of σ acting on $\bar{X}_i$.

Let $m = \mathrm{ord}(\sigma) \in \mathrm{Aut}(\bar{X}_i)$. Let $d := \gcd(m, n_i)$. Any lift of σ to $\bar{Y}_i$ satisfies $\sigma^*(\bar{y}_i) = \gamma \bar{y}_i / g(\bar{x}_i)$, where γ satisfies $\gamma^{n_i} = c_i$. It follows that γ, and hence σ, has order dividing $\mathrm{lcm}(m, n_i)$. It follows that the order of $G_i \cap I_{\bar{Y}_i}$ divides d. □

Remark 5.3. If $G_i \cap I_{\bar{Y}_i} = G_i$, the curve $\bar{Y}_i/G_i \cap I_{\bar{Y}_i}$ has genus 0. In this case, the component does not contribute to the local L-factor, and we may ignore it. This happens reasonably often, especially if n_i is prime. The following lemma explains that one can sometimes easily predict whether this happens. Recall that the action of $I_{\bar{X}_i}$ may immediately be deduced from the reduction of the branch points. To apply the lemma below it is therefore not necessary to calculate the action of inertia on $\bar{Y}_i$.

Lemma 5.4. *We use the notation of Lemma 5.2.*

(a) *Let* $\mu := \mathrm{ord}(c_i)$. *Assume that* $\delta := \gcd(\mu, n_i) \neq 1$. *Then the order of* σ *acting on* $\bar{Y}$ *is divisible by* δm.

(b) *In particular, if* $m\delta = n_i$, *then* $G_i \cap I_{\bar{Y}_i} = G_i$, *and the component does not contribute to the local* L*-factor.*

Proof. Let $\mu := \mathrm{ord}(c_i)$. Recall that μ divides the order m of σ acting on $\bar{X}_i$. Assume that $\delta := \gcd(\mu, n_i) \neq 1$. Define γ as in the proof of Lemma 5.2, i.e. $\gamma^{n_i} = c_i$. It follows now that the order of γ, and hence of σ acting on $\bar{Y}_i$, is δm. This proves (a). Part (b) follows immediately from (a) and Lemma 5.2.(b). □

Example 5.5. Let $\bar{Y}$ be the smooth projective curve over k given by

$$\bar{y}^n = \bar{x}(\bar{x}^2 + 1) =: \bar{f}(\bar{x}).$$

We assume that $\sigma(\bar{x}) = -\bar{x}$, then $\sigma^*\bar{f} = -\bar{f}$. If n is even, any lift of σ to $\bar{Y}$ satisfies

$$\sigma(\bar{x}, \bar{y}) = (-\bar{x}, \zeta\bar{y}),$$

where ζ is a $2n$th root of unity with $\zeta^n = -1$. It follows that the order of σ is divisible by 4.

In the case that $n = 4$ it follows that the order of σ is $8 = 2 \cdot n_i$. We conclude that $G_i \subset I_i$, and hence that $g(\bar{Y}_i/I_i) = 0$.

If n is odd, there exists an order-2 lift of σ to $\bar{Y}$, namely $\sigma(\bar{x}, \bar{y}) = (-\bar{x}, -\bar{y})$. A general lift satisfies

$$\sigma(\bar{x}, \bar{y}) = (-\bar{x}, -\zeta\bar{y}),$$

where ζ is a nth root of unity. In this case this information does not suffice to determine whether $g(\bar{Y}/I_{\bar{Y}}) = 0$, and we need to compute σ by the method explained in § 4.6.

Step II: Dividing out by P_i. We may assume that $G_i \cap I_{\bar{Y}_i} = \{1\}$. We consider the Sylow p-subgroup P_i of $I_{\bar{Y}_i}$. Since $I_{\bar{Y}_i}$ commutes with G_i, we may regard P_i as a subgroup of $\mathrm{Aut}_k(\bar{X}_i)$. Since $g(\bar{X}_i) = 0$, this is an elementary abelian p-group. We may normalize the coordinate on $\bar{X}_i$ such that the unique fixed point of P_i is $\bar{x}_i = \infty$. It follows that P_i may be identified with an $\mathbb{F}_p$-subspace $V_i \subset (k, +)$. Concretely, we may write

$$P_i = \{\sigma_a : \bar{x}_i \mapsto \bar{x}_i + a \mid a \in V_i\}.$$

It follows that the parameter

$$\bar{w}_i := \prod_{a \in V_i} (\bar{x}_i + a)$$

is invariant under P_i. (Note that $\bar{w}_i$ is an additive polynomial in $\bar{x}_i$.)

To compute the invariants of P_i, we need to rewrite the Kummer equation (19) in terms of the new parameter $\bar{w}_i$. Note that we may apply Lemma 5.2 in this situation, as well. Since 1 is the only pth root of unity in k, it follows that $\sigma_a^*\bar{f}_i = \bar{f}_i$ for all $a \in V_i$. (Here we use that $p \nmid n$.) Since we assume that

$G_i \cap I_{\bar{Y}_i} = \{1\}$, it follows that σ_a lifts to $\bar{Y}_i$ as an element of order p. (This follows as in the proof of Lemma 5.2.) In particular, it is easy to see that

$$\sigma_a(\bar{x}_i, \bar{y}_i) = (\bar{x}_i + a, \bar{y}_i).$$

Galois theory implies that there exists a polynomial $\bar{h}_i$ such that

$$\bar{f}_i(\bar{x}_i) = \bar{h}_i(\bar{w}_i).$$

It follows that $\bar{Y}_i/P_i \to \bar{X}/P_i$ is given by the Kummer equation

$$\bar{y}_i^{n_i} = \bar{h}_i(\bar{w}_i).$$

Remark 5.6. It is possible that $\bar{Y}_i/P_i$ has genus zero. In this case the contribution to the local L-factor is trivial, and we may stop the calculation.

Step III: We may additionally assume that $P_i = \{1\}$. We are now reduced to the case that $I_{\bar{Y}_i}$ is cyclic of order prime to p and $G_i \cap I_{\bar{Y}_i} = \{1\}$. Let σ be a generator of $I_{\bar{Y}_i}$. Lemma 5.2 together with our assumptions imply that

$$\sigma^* \bar{f}_i = c_i \frac{\bar{f}_i}{\bar{g}_i^m}, \tag{20}$$

with $\gcd(\mathrm{ord}(c_i), n) = 1$. Moreover, it follows that the order of σ on $\bar{Y}_i$ equals the order of σ on $\bar{X}_i$. (If not, some power of σ would be a nontrivial element of G_i.)

Let $m := \mathrm{ord}(\sigma)$. The inertia generator σ has two fixed points on $\bar{X}_i$.

Case a: We first assume that the fixed points of σ are k-rational. Then we may normalize the coordinate $\bar{x}_i$ of $\bar{X}_i$ such that the fixed points are $\bar{x}_i = 0, \infty$. It follows that $\sigma \bar{x}_i = \zeta_m \bar{x}_i$ for some primitive mth root of unity. Note that $\gcd(m, n_i) = 1$ in our situation. Moreover, the polynomial $\bar{g}_i$ in (20) is trivial. Together with the normalization of $\bar{x}_i$, this implies that we may write

$$\bar{f}_i(\bar{x}_i) = \bar{x}_i^{\alpha} \bar{\phi}_i(\bar{x}_i^m).$$

We deduce that $\sigma \bar{y}_i = \zeta_m^{\beta} \bar{y}_i$, where

$$\beta n_i \equiv \alpha \pmod{m}.$$

(Note that β exists, since $\gcd(m, n_i) = 1$.)

The new parameters

$$\bar{w}_i = \bar{x}_i^m, \qquad \bar{z}_i := \bar{y}_i \bar{x}_i^{-\beta}$$

are invariant under σ. We conclude that

$$\bar{z}_i^{n_i} = \bar{w}_i^{(\alpha - \beta n_i)/m} \bar{\phi}_i(\bar{w}_i)$$

is an equation for $\bar{Y}_i/I_{\bar{Y}_i} \to \bar{X}_i/I_{\bar{X}_i}$. This computes the invariants in Case (a).

Case b: Assume that the two fixed points of σ are defined over the quadratic extension k' of k. We write $\alpha, \alpha' \in k'$ for the fixed points of σ. Note that

these are not branched in $\bar{Y} \to \bar{X}$, since the branch points of that map are assumed to be k-rational.

Choosing

$$\bar{z}_i := \frac{\bar{x}_i - \alpha}{\bar{x}_i - \alpha'} \tag{21}$$

as new coordinate on $\bar{X}$, we may write $\sigma(\bar{z}_i) = \zeta \bar{z}_i$, where $\zeta \in k'$ is some primitive mth root of unity. Over k', we may apply a reasoning similar to the previous case. We remark that the polynomial $\bar{g}_i$ in (20) with $\bar{x}_i$ replaced by $\bar{z}_i$ is nontrivial in this situation.

We conclude that the cover $\bar{\phi}$ is given by a Kummer equation

$$\bar{y}_i^{n_i} = \frac{\bar{f}_i(\bar{z}_i)}{\bar{g}_i(\bar{z}_i)} = \frac{\bar{\phi}_i(\bar{z}_i^m)}{(\bar{z}_i - 1)^M}.$$

Here $M = \sum_{i=0}^r a_i$, where the a_i are as defined in § 4.1. We are now in a situation that the zeros of $\bar{f}_i$ are rational, therefore the irreducible factors of $\bar{f}_i$ have degree 1. Note that $\bar{z}_i = 0, \infty$ (which correspond to $\bar{x}_i = \alpha, \alpha'$) are not zeros of $\bar{f}_i$, since the zeros of $\bar{f}_i$ are k-rational. The factor $\bar{z}_i - 1$ is the denominator of the change of coordinates inverse to (21), which is

$$\bar{x}_i = \frac{-\alpha' \bar{z}_i - \alpha}{\bar{z}_i - 1}.$$

With respect to the coordinates $\bar{w}_i := \bar{y}_i(\bar{z}_i - 1)^{N/n}$ and $\bar{z}_i$, we obtain again a Kummer equation

$$\bar{w}_i^n = \bar{f}_i(\bar{z}_i) = \bar{\phi}_i(\bar{z}_i^m).$$

Note that $n_i \mid M$, by definition. It follows that

$$\sigma(\bar{z}_i, \bar{w}_i) = (\zeta \bar{z}_i, \zeta^\beta \bar{w}_i),$$

for some β. Since the order of σ is m, it follows that the order of ζ^β divides $\gcd(n, m)$. Since ζ is an mth root of unity, it follows that βn is divisible by m.

Now we are now in a similar situation as in Case a, but this time over the field k'. Setting

$$\bar{w}_i' = \bar{w}_i \bar{z}_i^{-\beta}, \qquad \bar{z}_i' := \bar{z}_i^m$$

yields the invariant equation

$$(\bar{w}_i')^{n_i} = (\bar{z}_i')^{-\beta n_i/m} \bar{\phi}_i(\bar{z}_i').$$

It remains to descend this model to k.

To find the correct model of our cover over k, we write $\rho \in \mathrm{Gal}(k'/k)$ for the generator of $\mathrm{Gal}(k'/k)$. By definition, ρ permutes α and α'. One computes that ρ induces the following action on the chosen coordinates:

$$\begin{aligned}
&\rho^*(\bar{x}_i) = \bar{x}_i, \quad \rho^*(\bar{y}_i) = \bar{y}_i,\\
&\rho^*(\bar{z}_i) = \bar{z}_i^{-1}, \quad \rho^*(\bar{w}_i) = \bar{w}_i/(-\bar{z}_i)^{N/n}\\
&\rho^*(\bar{z}_i') = \bar{z}_i'^{-1}, \quad \rho^*(\bar{w}_i') = (-1)^N (\bar{w}_i' \bar{z}_i')^{(\beta n_i - N)/m}.
\end{aligned}$$

It follows that the invariants of ρ are

$$\bar{z}_i' + (\bar{z}_i')^{-1}, \qquad \bar{w}_i' + (-1)^N \bar{w}_i' \bar{z}^{(\beta n_i - N)/m}.$$

They satisfy the equation

$$(\bar{w}_i' + (-1)^N \bar{w}_i' \bar{z}^{(\beta n_i - N)/m})^{n_i} = (\bar{z}_i')^{-\beta n_i/m}(1 + (-1)^N \bar{z}^{(\beta n_i - N)/m})^{n_i} \bar{\phi}_i(\bar{z}_i').$$

By construction, the right hand side of this equation is a polynomial in $\bar{z}_i' + (\bar{z}_i')^{-1}$.

Example 5.7. Suppose that $\bar{\phi} : \bar{Y}_i \to \bar{X}_i$ over $k = \mathbb{F}_p$ is given by

$$\bar{y}^3 = \bar{x}(\bar{x} - 1).$$

We assume that $p \equiv 2 \pmod 3$. We consider the automorphism

$$\sigma(\bar{x}, \bar{y}) = \left(\frac{-1}{x+1}, \frac{-y}{x+1} \right).$$

Over $\mathbb{F}_{p^2}$, we may choose a primitive 3rd root of unity ζ, and define the new coordinates

$$\bar{z} = \frac{\bar{x} - \zeta}{\bar{x} - \zeta^2}, \quad \bar{w} = \bar{y}(\bar{z} - 1).$$

With respect to these new coordinates, the automorphism σ is given by

$$\sigma(\bar{z}, \bar{w}) = (\zeta^2 \bar{z}, \zeta \bar{w}).$$

Moreover, the cover ϕ is given by

$$\bar{w}^3 = \bar{z}^3 - 1.$$

We define the invariants $\bar{z}' = z^3$ and $\bar{w}' = w \cdot z$, and find the invariant equation

$$(\bar{w}')^3 = \bar{z}'(\bar{z}' - 1).$$

Let ρ be the generator of $\mathrm{Gal}(\mathbb{F}_{p^2}, \mathbb{F}_p)$. Then

$$\rho(\bar{z}') = \frac{1}{\bar{z}'}, \quad \rho(\bar{w}') = \frac{-\bar{w}'}{\bar{z}'}.$$

We find as invariants of ρ:

$$\bar{u} := \bar{z}' + (\bar{z}')^{-1}, \quad \bar{v} := \bar{w}' - \bar{w}'/\bar{z}'.$$

These invariants satisfy the equation

$$\bar{v}^3 = (\bar{w}')^3 \left(1 - \frac{1}{\bar{z}'} \right)^3 = \frac{(\bar{z}' - 1)^4}{(\bar{z}')^2} = \bar{u}^2 - 4.$$

This is the equation of the invariants over $\mathbb{F}_p$ we wanted to compute.

5.2. Descent. We now describe the final step: computing the correct k-model of $\bar{Y}/I_K$. This amounts to computing the invariants of

$$\mathrm{Gal}(L,K)/(I_K \cap \mathrm{Gal}(L,K)) \simeq \Gamma_K/I_K$$

acting on $\bar{Z} = \bar{Y}/I_K$. We recall the definition of L from § 4.2 and 4.3.

The field L_0 was defined as the splitting field of f over K^{nr} (§ 4.2). In fact, a careful analysis of our algorithm shows that it is not necessary to pass to K^{nr}. It suffices to adjoin the $n$ths roots of unity. We may therefore redefine L_0 as the splitting field of f over $K(\zeta_n)$. The field extension L/L_0 is described in Proposition 4.4. The degree $[L : L_0]$ divides n.

We start by describing the correct model of $\bar{X}/I_K$ over $\mathbb{F}_q$ (which is the residue field of K). It follows from the above description, that it suffices to describe the action of $\mathrm{Gal}(k_0/\mathbb{F}_q)$ on $\bar{X}/I_K$. Here k_0 denotes the residue field of L_0. The quotient of $\mathrm{Gal}(k_0/\mathbb{F}_q)$ which acts nontrivially on $\bar{X}/I_K$ is a quotient of the Galois group of the splitting field of f over K. Therefore this action is completely determined by the action on the specialization of the branch points (i.e. the roots of f). In particular, it is not necessary to compute the action from the definition of the coordinates on the components of $\bar{X}$ using the method explained in § 4.6.

Next we need to describe the action of $\mathrm{Gal}(L/K)$ on $\bar{Z}$. Adjoining the nth roots of unity only effects the Galois action, but not the model of the curve $\bar{Y}$, therefore we may ignore this action. The action of $\mathrm{Gal}(L_0, K)$ is also determined by the action on the specialization of the ramification points. Therefore it suffices to determine the action of $\mathrm{Gal}(L, L_0)$. In § 5.1, we have seen that $\bar{Z} \to \bar{X}/I_K$ is still given by a Kummer equation, therefore this action may be described as in the proof of Proposition 4.4.

In many cases it is clear that $\mathrm{Gal}(L, L_0)$ acts trivially on $\bar{Z}$, and it suffices to consider the action of the Galois group on the specialization of the ramification points (compare to Remark 5.3). An example of this situation can be found in § 6.

6. An example

6.1. We consider the Kummer cover $\phi : Y \to X = \mathbb{P}^1_K$ over $K := \mathbb{Q}_3$ given by the equation

$$y^4 = f(x) = (x^2 - 3)(x^2 + 3)(x^2 - 6x - 3).$$

The ramification points of ϕ are the six root of f (with ramification index 4) and the point at ∞ (with ramification index 2). The Riemann–Hurwitz formula shows that the genus of Y is 7.

The splitting field of f over $K^{\mathrm{nr}} = \mathbb{Q}_3^{\mathrm{nr}}$ is $L_0 := K^{\mathrm{nr}}(3^{1/2})$, and the roots of f are

$$\pm 3^{1/2}, \pm i 3^{1/2}, \alpha, \alpha',$$

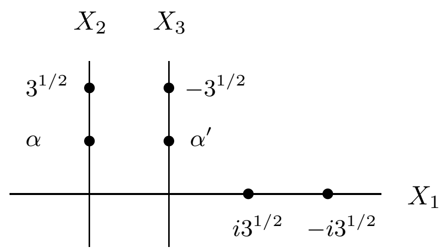
where $i \in K^{\rm nr}$ is a fourth root of unity and $\alpha = 3 - 2\cdot 3^{1/2}, \alpha' = 3 + 2\cdot 3^{1/2} \in L_0$ are the two roots of $x^2 - 6x - 3$. Note that

$$\alpha \equiv 3^{1/2}, \quad \alpha' \equiv -3^{1/2} \pmod 3.$$

Let L/L_0 be a tamely ramified extension. Following the procedure described in § 4.3, it is easy to see that the stably marked model $(\mathcal{X}, \mathcal{D})$ of (X_L, D_L) is given by the three charts $\lambda_i : X_L \to \mathbb{P}^1_L$, $i = 1, 2, 3$ corresponding to the parameters

$$x_1 := 3^{-1/2} x, \quad x_2 := \frac{x - 3^{1/2}}{3}, \quad x_3 := \frac{x + 3^{1/2}}{3}.$$

Let $X_i \subset \bar{X} := \mathcal{X} \otimes k$ be the irreducible component corresponding to λ_i. Then $\bar{X}$ looks as follows:



In this picture the dots indicate the position of a point $\bar{\alpha}_i \in \bar{D} \subset \bar{X}$. Next to the dots one finds the value of the corresponding point $\alpha_i \in D_L \subset X_L = \mathbb{P}^1_L$.

6.2. Let $\mathcal{Y}$ denote the normalization of $\mathcal{X}$ in the function field of Y_L. If we choose L/L_0 sufficiently large, $\mathcal{Y}$ is semistable (Corollary 3.5). Using the arguments of § 4.5 (in particular Proposition 4.4) we can see how large L has to be and compute the special fiber $\bar{Y} := \mathcal{Y} \otimes k$.

Let $N := [L : K^{\rm nr}]$ denote the absolute ramification index of L. By assumption, N is prime to p. Therefore, $L = K^{\rm nr}[3^{1/N}]$, where $3^{1/N}$ is some Nth root of 3. Moreover, $3^{1/N}$ is a prime element of $\mathcal{O}_L$. Let η_i denote the normalized discrete valuation on $L(X)$ corresponding to the component X_i and set $N_i := \eta_i(f)$. Furthermore, we set $f_i := 3^{-N_i/N} f$ and let $\bar{f}_i$ denote the image of f_i in the residue field $k(\eta_i) = k(\bar{x}_i)$. By Proposition 4.4, the knowledge of N_i and $\bar{f}_i$ for $i = 1, 2, 3$ is sufficient to determine the stable reduction $\bar{Y}$ of Y.

For $i = 1$ we write

$$f(x) = f(3^{1/2} x_1) = 3^3 (x_1^2 - 1)(x_1^2 + 1)(x_1^2 - 2 \cdot 3^{1/2} x_1 - 1),$$

from which we conclude that

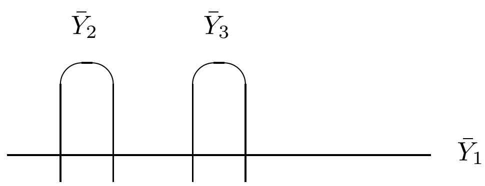
$$\eta_1(f) = 3, \quad \bar{f}_1 = (\bar{x}_1^2 - 1)^2 (\bar{x}_1^2 + 1).$$

Similarly, we check that for $i = 2, 3$ we have

$$\eta_i(f) = 4, \quad \bar{f}_i = 2x_i(\bar{x}_i - 1).$$

By the first part of Proposition 4.4 it follows that $\mathcal{Y}$ is semistable if and only if $4|N$, i.e. L contains a 4th root of 3. Furthermore, if we assume $4|N$ then by the second part of the proposition there is a unique component $\bar{Y}_i$ of $\bar{Y}$ lying above $\bar{X}_i$, which is smooth, and $Y_i \to X_i$ is the Kummer cover with equation $\bar{y}_i^4 = \bar{f}_i$, for $i = 1, 2, 3$. Note that the genus of $\bar{Y}_1$ is equal to 3, while the genus of $\bar{Y}_2$ and $\bar{Y}_3$ is equal to 1.

To describe $\bar{Y}$ we still need to understand how the components $\bar{Y}_i$ intersect each other. By Remark 3.6 (2), the singular locus of $\bar{Y}$ is precisely the inverse image of the singular locus of $\bar{X}$. The latter is contained in the component $\bar{X}_1$, and consists of the two points with $\bar{x}_1 = \pm 1$. But these are precisely the branch points of the cover $\bar{Y}_1 \to \bar{X}_1$ with branching order 2. It follows that the singular locus of $\bar{Y}$ is contained in $\bar{Y}_1$ and consists of the 4 ramification points of order 2. Two of these points are contained in $\bar{Y}_2$ (resp. in $\bar{Y}_3$). Therefore, the picture of $\bar{Y}$ looks as follows.



We see that the arithmetic genus of $\bar{Y}$ is equal to $3 + 1 + 1 + 2 = 7$, which is equal to the genus of Y, as it should be.

6.3. We now look at the action of the inertia group I_K on $\bar{Y}$. This action factors over the tame quotient $\mathrm{Gal}(K^{\mathrm{nr}}(3^{1/4})/K^{\mathrm{nr}})$ of order 4. Let σ be the generator with $\sigma(3^{1/4}) = i \cdot 3^{1/4}$.

The action of I_K on the stably marked curve $(\bar{X}, \bar{D})$ is determined by its action on the set D_L. In fact, the element σ acts as an involution on this set, as follows:

$$3^{1/2} \leftrightarrow -3^{1/2}, \quad i3^{1/2} \leftrightarrow -i3^{1/2}, \quad \alpha \leftrightarrow \alpha'.$$

It follows that σ maps the component $\bar{X}_1$ of $\bar{X}$ to itself and interchanges the two components $\bar{X}_2, \bar{X}_3$. Also, σ^2 is the identity on $\bar{X}$. It follows that σ fixes the component $\bar{Y}_1$ of $\bar{Y}$ and interchanges the two components $\bar{Y}_2, \bar{Y}_3$.

The restriction of σ to $\bar{X}_1$ is given by $\sigma(\bar{x}_1) = -\bar{x}_1$. The coordinate $\bar{y}_1$ is the image in $k(\bar{Y}_1)$ of $y_1 := 3^{-3/4}y$. It follows that

$$\sigma(\bar{y}_1) = i\bar{y}_1$$

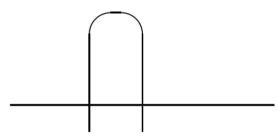
(compare to § 4.6).

Let $Z_1 := Y_1/I_K$ denote the quotient of Y_1 by the inertia action. The function field of Z_1 is the fixed field

$$k(\bar{x}_1, \bar{y}_1)^{<\sigma>} = k(\bar{z}_1), \quad \text{with } \bar{z}_1 := \frac{\bar{y}_1^2}{\bar{x}_1(\bar{x}_1^2 - 1)}.$$

In particular, $Z_1 \cong \mathbb{P}^1_k$ has genus zero.

By a similar analysis we see that $\sigma(\bar{x}_2) = \bar{x}_3$, $\sigma(\bar{x}_3) = \bar{x}_2$ and $\sigma^2(\bar{y}_j) = \bar{y}_j$ for $j = 2, 3$. It follows that σ interchanges the two components $\bar{Y}_2$ and $\bar{Y}_3$ and that σ^2 is the identity on $\bar{Y}_2 \cup \bar{Y}_3$. All in all we see that the quotient curve $\bar{Z} := barY/I_K$ is a semistable curve consisting of two components, $\bar{Z}_1$ and $\bar{Z}_2$, where $\bar{Z}_1 = \bar{Y}_1/I_K$ is as above and $\bar{Z}_2 := (\bar{Y}_2 \cup \bar{Y}_3)/I_K$ is an isomorphic copy of $\bar{Y}_2$ (or $\bar{Y}_3$). The two components intersect each other in two points, as follows.



The arithmetic genus of $\bar{Z}$ is equal to $g(\bar{Z}) = g(\bar{Z}_1) + g(\bar{Z}_2) + 1 = 0 + 1 + 1 = 2$.

6.4. It remains to determine the semilinear action of $\Gamma_k = \Gamma_K/I_K$ on $\bar{Z}$. Doing this is equivalent to describing a corresponding $\mathbb{F}_3$-model $\bar{Z}_{\mathbb{F}_3}$ of $\bar{Z}$. More practically, this is the same as finding generators of the extension of function fields $k(\bar{Y}_i)/k$ which are Γ_k-invariant, for $i = 1, 2, 3$.

Let $\bar{\sigma} \in \Gamma_k$ be an arbitrary element. We lift $\bar{\sigma}$ to an element $\sigma \in \Gamma_K$. The same computation as above (when we analyzed the action of I_K on $\bar{Y}$) shows that

$$\sigma(\bar{y}_1) = \psi(\sigma) \cdot \bar{y}_1, \quad \sigma(\bar{x}_1) = \psi(\sigma)^2 \cdot \bar{x}_1,$$

where $\psi(\sigma) \in \mu_4(k)$ is defined as the image of $\sigma(3^{1/4})/3^{1/4} \in \mathcal{O}_L^\times$ in k. It follows immediately that

$$\bar{\sigma}(\bar{z}_1) = \sigma\big(\frac{\bar{y}_1^2}{\bar{x}_1(\bar{x}_1^2 - 1)}\big) = \bar{z}_1.$$

We conclude that the natural $\mathbb{F}_3$-model of $\bar{Z}_1$ is isomorphic to $\mathbb{P}^1_{\mathbb{F}_3}$.

Similarly, one sees that an element $\sigma \in \Gamma_K$ either permutes the functions $\bar{x}_2, \bar{x}_3, \bar{y}_2, \bar{y}_3$ as follows,

$$\bar{x}_2 \leftrightarrow \bar{x}_3, \quad \bar{y}_2 \leftrightarrow \bar{y}_3,$$

or fixes each of them (here we think of these as elements of $k(\bar{Y}_2) \oplus k(\bar{Y}_3)$, the 'function ring' of $\bar{Y}_2 \cup \bar{Y}_3$). The latter happens if and only if σ fixes $3^{1/2}$. So the function field of $\bar{Z}_2 = (\bar{Y}_2 \cup \bar{Y}_3)/I_K$ can be identified with the fixed ring

$$(k(\bar{Y}_2) \oplus k(\bar{Y}_3))^{I_K} = k(\bar{x}, \bar{y}),$$

where $\bar{x} := \bar{x}_2 + \bar{x}_3$ and $\bar{y} := \bar{y}_2 + \bar{y}_3$. It follows that $\bar{Z}_{2,\mathbb{F}_3}$ is the smooth projective curve of genus one over $\mathbb{F}_3$ with equation

$$\bar{y}^4 = 2\bar{x}(\bar{x} - 1).$$

To complete our description of $\bar{Z}_{\mathbb{F}_3}$ we have decide how Γ_k acts on the singular locus of $\bar{Z}$. As a subset of $\bar{Z}_1 \cong \mathbb{P}^1_k$ it consists of the image under $\bar{Y}_1 \to \bar{Z}_1$ of the four points with $\bar{x}_1^2 = 1$. Using the relation $\bar{z}_1^2 = (\bar{x}_1^2+1)/\bar{x}_1^2$, we see that the singular locus of $\bar{Z}$ consists of the two points of $\bar{Z}_1 \cong \mathbb{P}^1_k$ with $\bar{z}_1^2 = 2$. But we have shown above that $\bar{z}_1$ is Γ_k-invariant. It follows that the singular locus of $\bar{Z}_{\mathbb{F}_3}$ consists of two geometric points which are conjugate over the quadratic extension $\mathbb{F}_9/\mathbb{F}_3$. This completes our description of $\bar{Z}_{\mathbb{F}_3}$.

6.5. We can now write down the local L-factor for the curve $Y/\mathbb{Q}_3$. By Corollary 2.5, the local factor is

$$L_3(Y,s) = P_1(\bar{Z}, 3^{-s}),$$

where

$$P_1(\bar{Z},T) := \det\big(1 - \mathrm{Frob}_3 \cdot T | H^1(\bar{Z},\mathbb{Q}_\ell)\big)$$

and where $\mathrm{Frob}_3 : \bar{Z}_{\mathbb{F}_3} \to \bar{Z}_{\mathbb{F}_3}$ is the $\mathbb{F}_3$-Frobenius endomorphism.

The normalization of $\bar{Z}$ is equal to the disjoint union of $\bar{Z}_1 \cong \mathbb{P}^1_k$ and $\bar{Z}_2$. Hence the short exact sequence (15) takes the form

$$0 \to H^1(\bar{Z}_2,\mathbb{Q}_\ell) \to H^1(\bar{Z},\mathbb{Q}_\ell) \to H_1(\Delta_{\bar{Z}},\mathbb{Q}_\ell) \to 0.$$

It leads to a decomposition of $P_1(\bar{Z},T)$ as a product of two polynomials, of degree one and two, respectively.

The first factor is determined by the action of Frob_3 on $H_1(\Delta_{\bar{Z}},\mathbb{Q}_\ell)$. We have an isomorphism $H_1(\Delta_{\bar{Z}},\mathbb{Q}_\ell) \cong \mathbb{Q}_\ell$, canonical up to sign. By the description of the Γ_k-action on $\bar{Z}$ from the previous subsection we see that Frob_3 fixes the two components $\bar{Z}_1$ and $\bar{Z}_2$ of $\bar{Z}$, while it interchanges the two singular points. It follows that Frob_3 acts on $H_1(\Delta_{\bar{Z}},\mathbb{Q}_\ell)$ by multiplication with -1. Hence the first factor of $P_1(\bar{Z},T)$ is equal to

$$1 + T.$$

The second factor is simply the numerator of the local zeta function of the genus-one curve

$$\bar{Z}_{2,\mathbb{F}_3} : \quad \bar{y}^4 = 2\bar{x}(\bar{x}-1).$$

It suffices to count the number of $\mathbb{F}_3$-rational points, which is

$$|\bar{Z}_{\mathbb{F}_3}(\mathbb{F}_3)| = 4.$$

It follows that the trace of Frob_3 on $H^1(Z,\mathbb{Q}_\ell)$ is equal to $a_p = p + 1 - |\bar{Z}_{\mathbb{F}_3}(\mathbb{F}_3)| = 0$. We conclude that

$$P_1(\bar{Z},T) = (1+T)(1+pT^2).$$

6.6. We can also use our description of the stable reduction of Y to compute the exponent of the conductor of the Γ_K-representation $H^1(Y_{\bar{K}},\mathbb{Q}_\ell)$. In fact, this exponent is of the form

$$f = \epsilon + \delta,$$

where

$$\epsilon := \dim H^1(Y_{\bar{K}},\mathbb{Q}_\ell) - \dim H^1(Y_{\bar{K}},\mathbb{Q}_\ell)^{I_K}$$

and where δ is the *Swan conductor* ([19], § 2). By Theorem 2.1 and the above computations we have

$$\epsilon = 2g(Y) - \dim H^1(\bar{Z}, \mathbb{Q}_\ell) = 14 - 3 = 11.$$

Since Y achieves semistable reduction over a tame extension of $\mathbb{Q}_3^{\mathrm{nr}}$, the Swan conductor vanishes. We conclude that the exponent of the conductor is equal to

$$f = 11.$$

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